

# DYNAMICS IN DUMBBELL DOMAINS I. CONTINUITY OF THE SET OF EQUILIBRIA

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ABSTRACT. We analyze the dynamics of a reaction-diffusion equation with homogeneous Neumann boundary conditions in the dumbbell domain. We provide an appropriate functional setting to treat this problem and, as a first step, we show in this paper the continuity of the set of equilibria and of its linear unstable manifolds.

## 1. INTRODUCTION

This paper is the first one of a series of articles whose final objective is to address the problem of the behavior of the asymptotic nonlinear dynamics of a reaction-diffusion equation when the domain where the equation is posed undergoes a singular perturbation.

In particular, we consider the evolution equation of parabolic type of the form

$$\begin{cases} u_t - \Delta u + u = f(u), & x \in \Omega_\epsilon, \ t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \quad (1.1)$$

where  $\Omega_\epsilon \subset \mathbb{R}^N$ ,  $N \geq 2$ , is a bounded smooth domain,  $\epsilon \in (0, 1]$  is a parameter,  $\frac{\partial}{\partial n}$  is the outside normal derivative and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a dissipative nonlinearity.

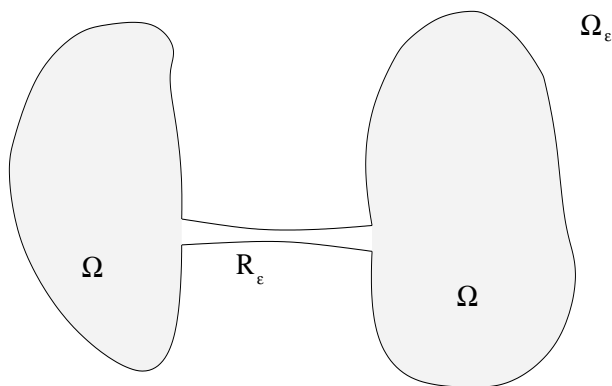


FIGURE 1. Dumbbell domain

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The domain  $\Omega_\epsilon$  is a dumbbell type domain consisting of two disconnected domains, that we will denote by  $\Omega$ , joined by a thin channel,  $R_\epsilon$ , which degenerates to a line segment as the parameters  $\epsilon$  approaches zero, see Figure 1.

Under standard dissipative assumption on the nonlinearity  $f$  of the type,  $\limsup_{|s| \rightarrow +\infty} f(s)/s < 1$ , for fixed  $\epsilon \in (0, 1]$  equation (1.1) has an attractor  $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$ .

On the other hand, passing to the limit as  $\epsilon \rightarrow 0$ , the limit “domain” will consist of the domain  $\Omega_0$  and a line in between, see Figure 2

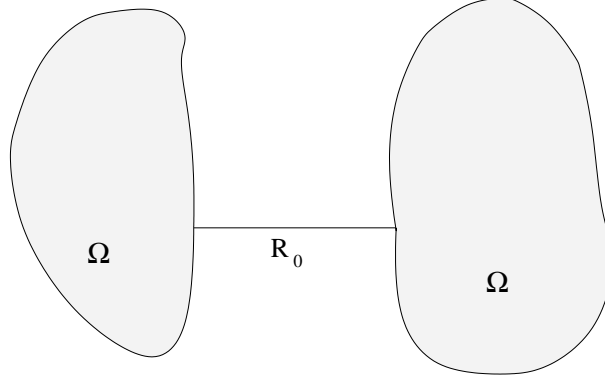


FIGURE 2. Limit “domain”

And the limit equation is

$$\begin{cases} w_t - \Delta w + w = f(w), & x \in \Omega, \ t > 0 \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ v_t - Lv + v = f(v), & s \in R_0 \\ v(p_0) = w(p_0), \ v(p_1) = w(p_1) \end{cases} \quad (1.2)$$

where  $w$  is a function that lives in  $\Omega$  and  $v$  lives in the line segment  $R_0$ . Moreover,  $L$  is a differential operator which depends on the geometry of the channel  $R_\epsilon$ , more exactly, on the way the channel  $R_\epsilon$  collapses to the segment line  $R_0$ . For instance, in two dimensions, if the channel  $R_\epsilon = \{(x, y) : 0 < x < 1, 0 < y < \epsilon g(x)\}$ , then  $Lv = \frac{1}{g}(gv_x)_x$ . For other channels, the operator  $L$  needs to be calculated explicitly. We also denote by  $p_0$  and  $p_1$  the points where the line segment touches the domain  $\Omega$ . Again, this system has an attractor  $\mathcal{A}_0$  in  $H^1(\Omega) \times H^1(R_0)$ .

We are interested in understanding the relation between the attractors  $\mathcal{A}_\epsilon$ ,  $\epsilon \in (0, 1]$  and  $\mathcal{A}_0$ . With the results of this paper and with [7], we will show that this family of attractors is upper semicontinuous at  $\epsilon = 0$  in certain topology, and if all the equilibria in  $\mathcal{A}_0$  are hyperbolic, then the attractors are continuous, that is, upper and lower semicontinuous.

In appropriate functional spaces, we will see that problem (1.1) can be written as an evolutionary equation of the type

$$\begin{cases} u_t + A_\epsilon u = F_\epsilon(u), & t > 0 \\ u(0) \in X_\epsilon \end{cases} \quad (1.3)$$

for certain family of spaces  $X_\epsilon$ . Also, problem (1.2) can also be written as

$$\begin{cases} u_t + A_0 u = F_0(u), & t > 0 \\ u(0) \in X_0 \end{cases} \quad (1.4)$$

in a certain space  $X_0$ .

In this paper, we will work out an appropriate functional setting to treat a broad class of perturbation problems which, in particular, will include the case of the singular perturbation problem of the dumbbell domain, that is, problems (1.3) and (1.4). This functional setting will make use of several concepts like the concept of convergence for a sequence  $\{u_\epsilon\}_{\epsilon \in (0,1]}$  where  $u_\epsilon$  belongs to different spaces  $X_\epsilon$  for each  $\epsilon \in (0,1]$ , an appropriate concept of compactness for families living in different spaces and the concept of “compact convergence” as the key concept to treat the behavior of compact operators in different spaces. This setting is developed mainly in sections 4 and 5.

The program that we will follow to prove the continuity of the attractors is divided in two parts. The first one, which is addressed in this paper, consists in proving the continuity of the equilibria and, in case the equilibrium is hyperbolic, obtaining the continuity of its linear unstable manifolds. Hence:

- (1) We will first show the convergence of the resolvent operators, that is will show that  $A_\epsilon^{-1}$  converge in an appropriate way to  $A_0^{-1}$ , see Proposition 2.7. This is a key point to all the analysis.
- (2) Writing the stationary problem as a fixed point problem, that is,  $u_\epsilon$  is an equilibrium solution of (1.3) (resp.  $u_0$  is an equilibrium of (1.4)) if  $u_\epsilon = A_\epsilon^{-1} F_\epsilon(u_\epsilon)$  (resp.  $u_0 = A_0^{-1} F_0(u_0)$ ) and with the convergence of the linear resolvent operators, we will show the convergence of the equilibria. Moreover, we will show that if an equilibrium of the limit problem is hyperbolic, then it is isolated and there exists one and only one equilibrium of the perturbed problem nearby, see Theorem 2.3.
- (3) With the convergence of the resolvent operators and with the convergence of the equilibria, we will also show the convergence of the resolvent operators of the linearizations around the equilibria, that is the convergence of  $(A_\epsilon - F'_\epsilon(u_\epsilon) + \lambda)^{-1}$  to  $(A_0 - F'_0(u_0) + \lambda)^{-1}$ , for some  $\lambda$  large enough. For the case where the equilibrium is hyperbolic, this will imply the convergence of the linear unstable manifolds, see Theorem 2.5.

The second part, which is developed in [7] consists in proving the convergence of the linear and nonlinear semigroups and the nonlinear unstable manifolds of the equilibria:

- (4) With the convergence of the resolvent operators  $A_\epsilon^{-1}$  to  $A_0^{-1}$  we will show, with a Trotter-Kato type formula, the convergence of the linear semigroups  $e^{-A_\epsilon t}$  to  $e^{-A_0 t}$ .
- (5) With the variation of constants formula we will show the convergence of the nonlinear semigroups. Once this is accomplished, the upper semicontinuity of attractors is easily obtained.

- (6) Assuming the equilibria are all hyperbolic, with the convergence of the linear unstable manifolds and with a similar argument as it is done in [6] we will be able to show the convergence of the local nonlinear unstable manifolds. Using now that the system is gradient we will easily show that the attractors are lower semicontinuous and therefore continuous.

This agenda, or variants of it, has been proved to be successful when addressing the behavior of the long time dynamics in different perturbation problems. It is based in a careful study of the behavior of the linear parts under the perturbation considered and this information is translated into the nonlinear dynamics through the variation of constants formula. In [5] a general approach to obtain upper semicontinuity of attractors following this approach is explained. Also, a similar technique to get the upper semicontinuity was used in [38] for the case of thin domains with “holes”. In [6, 1] this same technique is used to obtain the continuity (upper and lower semicontinuity) of the attractors of reaction diffusion equations with Neumann boundary conditions when the domain is perturbed. Actually, in this paper, the only condition imposed in the perturbed domains is the spectral convergence of the linear operators. Inspired by the works [6, 1] a general scheme to treat the continuity of the attractors of semilinear parabolic problems is developed in [9]. We also refer to [15, 17] for general theorems guaranteeing the lower semicontinuity of the attractors.

The “dumbbell domain” problem has been considered before by many authors. It appears naturally as the counterpart of a convex domain in the following situation. It is well known that in a convex domain the stable stationary solutions to (1.1) are necessarily spatially constant, see [10, 32]. This is due to the fact that gradients of temperature can be diffused in the shortest path (the line segment between the two points with different temperatures). One way to produce “patterns”, that is, stable stationary solutions which are not spatially constant, is to consider domains which makes it difficult for the heat to flow from one part of the domain to the other, making a constriction in the domain. It becomes natural to consider dumbbell like domains as a prototype domain to produce this “patterns”. With this purpose the so called dumbbell domains with a bistable nonlinearity of the type  $f(u) = u - u^3$  was considered in [35].

It seems clear that when passing from a convex domain to a non convex domain (like a dumbbell domain) some kind of bifurcation of equilibria appears. This aspect was studied in [19, 41].

In several works at the end of the 80’s ([22], [23], [24], [25], [26]) and beginning of the 90’s ([27], [28]) S. Jimbo made a detail analysis of the behavior of linear and semilinear elliptic problems in dumbbell type domains with two important characteristics: (1) the dimension is larger or equal to three and (2) the channel  $R_\epsilon$  is a straight cylindrical channel. His analysis is based in a very careful and detailed study of the  $L^\infty$  norm of the eigenfunctions of the Laplace operator with Neumann boundary conditions in the junction of the channel with the fixed part of the domain.

With regards to the spectral behavior of the Laplace operator in dumbbell domains we refer to [25] for a straight cylindrical channel and to [2, 3, 4] for more general channels. See also [16] for a survey on results on the behavior of eigenvalues under perturbations of the domain and [21] for a general method to treat regular perturbations of the domain. Recently

there has been a study of the rates of the eigenvalues of the dumbbell domains in dimension 3 with a cylindrical channel in [14]. Also, in [11] the authors analyze spectral properties in a multidimensional structure with similar properties as our limiting domain depicted in Figure 2.

In [31], S. Jimbo and Y. Morita made a detailed study of the first  $k$  eigenvalues of the Laplace operator with Neumann boundary conditions in a domain  $\Omega \subset \mathbb{R}^n$ , which consists of exactly  $k$  fixed subdomains joined by thin channels. These  $k$  eigenvalues approach zero and the  $k + 1$  eigenvalue is uniformly bounded away from zero. The thickness of each channel is controlled by a small parameter  $\varepsilon > 0$  and these channels approach a line segment connecting two subdomains in a certain sense (some of these channels may be empty). With the characterization of the first  $k$  eigenvalues and eigenfunctions for the operator  $-\Delta$  in this domain, in [30], the same authors apply the invariant manifold theory to show that the dynamics of an associated reaction diffusion problem with a nonlinearity such that its Lipschitz constant is small (compared in some concrete way to the  $k + 1$  eigenvalue), is equivalent to the dynamics of a system of coupled ordinary differential equation in the invariant manifold. The fact that the Lipschitz constant of the nonlinearity is small prevents in particular any contribution to the dynamics from the channel. We would also like to mention the work [36], which extended somehow the results of [30] and of [18].

In [28], S. Jimbo states a result on the continuity in the norm of the supremum of the attractors  $\mathcal{A}_\varepsilon$  for semilinear parabolic problems in dumbbell type domains where the channel connecting the two disjoint domains is a straight cylindrical one. But no proofs are given.

In [42] the author develops a functional framework to treat nonlinear elliptic problems on sequences of domains  $\{\Omega_n\}_{n=1}^\infty$ . The sequence of domains is assumed to be nested, all of them contain the limit domain,  $\Omega_0$ , and the sequence converges in measure to the limit domain. In this general context, the author obtains the upper semicontinuity of the set of equilibria. Moreover, under certain spectral convergence condition and certain restrictions on the Lipschitz constant of the nonlinearity, if the limit domain has a hyperbolic equilibrium, then for  $n$  large enough the equation has one and only one equilibrium nearby. The restriction on the Lipschitz constant of the nonlinearity is related to the restriction already mentioned in [31] and in particular it prevents from any contribution to the dynamics of the set  $\Omega_n \setminus \Omega_0$ . In particular, the results from this paper do not give information to the case of a dumbbell domain where the limit equation (1.2) has an equilibrium solution concentrated in the channel. This is the case for example if the channel is cylindrical and straight so that the operator  $L(v) = v''$ , the nonlinearity is  $f(u) = k(u - u^3)$  and  $k - 1$  is larger than the first eigenvalue of the Laplace operator with Dirichlet boundary conditions in the segment  $R_0$ .

For the formation of patterns in nonconvex domains for reaction-diffusion equations with nonlinear boundary conditions, we refer to [12, 13].

To the best of our knowledge the complete dynamical problem of a reaction-diffusion equation like (1.1) in a dumbbell domain in  $\mathbb{R}^N$  with  $N \geq 2$ , with the following characteristics:

- (1) the channel is not necessarily cylindrical,
- (2) there is no restriction in the Lipschitz constant of the nonlinearities, and
- (3) the limit equation (1.2) may have some dynamics in  $R_0$ , the limit of the thin channel,

has not been completely solved.

It is the purpose of this paper and of its continuation [7], to address this problem in its full generality.

This paper is organized as follows. In Section 2 we give the rigorous definition of the dumbbell domain, introduce some notation and state the main results of the paper, that is, the continuity of the set of equilibria and of its linear unstable manifold, Theorem 2.3 and Theorem 2.5. In this section we also state the basic result on the convergence of the resolvent of the linear operators, Proposition 2.7. In Section 3 we establish basic properties of the linear operators  $A_\epsilon$  and  $A_0$ . Section 4 is devoted to the abstract results using the notion of compact convergence that in particular will lead to the continuity of eigenvalues and eigenfunctions of the linear operators involved in the equations. The continuity of equilibrium solutions in a general setting is addressed in Section 5. We have also included, in Sections 4 and 5, several examples that show how we apply this general results to the case of the dumbbell domain. We give a proof of Theorem 2.3 and Theorem 2.5 in Section 6. Finally, we have included Appendix A, which is devoted to the proof of compact convergence of the resolvent in the case of dumbbell type domains, in particular, we show Proposition 2.7.

## 2. DEFINITION OF THE DOMAIN AND MAIN RESULTS

Before we can state in a precise way our main result, let us define the perturbation of the domain we are considering.

The family of domains we are dealing with is the so called dumbbell domain which, roughly speaking, consists of a pair of two fixed domains,  $\Omega$ , joined by a thin channel  $R_\epsilon$  which approaches a line segment as the parameter  $\epsilon$  approaches zero. In order to describe such domains we need to introduce some terminology.

Let  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , be a fixed open bounded and smooth domain such that there is an  $l > 0$  satisfying

$$\begin{aligned} \Omega \cap \{(s, x') : s^2 + |x'|^2 < l^2\} &= \{(s, x') : s^2 + |x'|^2 < l^2, s < 0\}, \\ \Omega \cap \{(s, x') : (s-1)^2 + |x'|^2 < l^2\} &= \{(s, x') : (s-1)^2 + |x'|^2 < l^2, s > 1\}, \\ \Omega \cap \{(s, x') : 0 < s < 1, |x'| < l\} &= \emptyset, \end{aligned}$$

with  $\{(0, x') : |x'| < l\} \cup \{(1, x') : |x'| < l\} \subset \partial\Omega$ . We are using the standard notation  $\mathbb{R}^N \ni x = (s, x')$ , with  $s \in \mathbb{R}$ ,  $x' = (x_2, \dots, x_N) \in \mathbb{R}^{N-1}$ .

The channel that we consider will be defined as  $R_\epsilon = \{(s, \epsilon x') : (s, x') \in R_1\}$  and  $R_1$  is defined as

$$R_1 = \{(s, x') : 0 \leq s \leq 1, x' \in \Gamma_1^s\}$$

and for all  $0 \leq s \leq 1$ ,  $\Gamma_1^s$  is diffeomorphic to the unit ball in  $\mathbb{R}^{N-1}$ . That is, we assume that for each  $s \in [0, 1]$ , there exists a  $C^1$  dipheomorphism

$$L_s : B(0, 1) \rightarrow \Gamma_1^s \tag{2.1}$$

Moreover, if we define

$$\begin{aligned} L : (0, 1) \times B(0, 1) &\rightarrow R_1 \\ (s, z) &\rightarrow (s, L_s(z)) \end{aligned} \tag{2.2}$$

then  $L$  is a  $C^1$  dipheomorphism.

Denote by  $g(s) := |\Gamma_1^s|$  the  $N-1$ -dimensional Lebesgue measure of the set  $\Gamma_1^s$ . From the smoothness of  $R_1$ , we may assume that  $g$  is a smooth function defined in  $[0, 1]$ . In particular, there exist  $d_0, d_1 > 0$  such that  $d_0 \leq g(s) \leq d_1$  for all  $s \in [0, 1]$ . Moreover, the channel  $R_\epsilon$  collapses to the line segment  $R_0 = \{(s, 0) : 0 \leq s \leq 1\}$

**Remark 2.1.** *A very important class of channels will be those whose transversal sections  $\Gamma_1^s$  are disks centered at the origin of radius  $r(s)$ , that is*

$$R_1 = \{(s, x'), |x'| < r(s), 0 \leq s \leq 1\}$$

*For this channel,  $g(s) = \omega_{N-1}r(s)^{N-1}$  where  $\omega_{N-1}$  is the Lebesgue measure of the unit ball in  $\mathbb{R}^{N-1}$ .*

*Many of the results in the literature are obtained for this particular channel, even for the completely straight channel given by  $r(s) \equiv 1$ , see for instance [22, 23, 24, 25].*

The dumbbell domain will be the domain  $\Omega_\epsilon = \Omega \cup R_\epsilon$  for  $\epsilon \in (0, 1]$ . Observe that we did not specify any connectedness property for  $\Omega$ . Therefore we can have the situation described in Figure 1 or as in Figure 3.

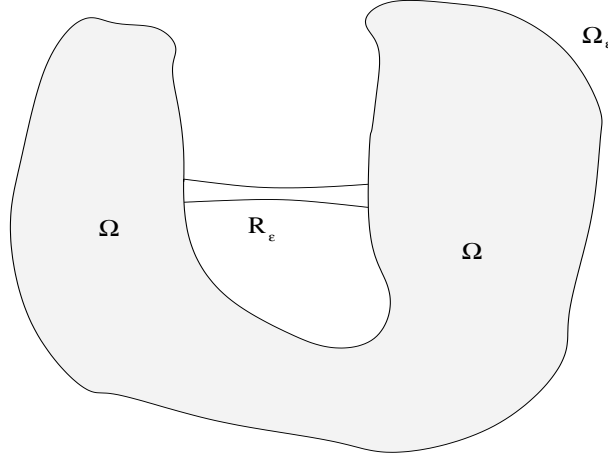


FIGURE 3. Dumbbell domain with a connected  $\Omega$

Consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u + u = f(u), & x \in \Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\epsilon. \end{cases} \quad (2.3)$$

defined in the dumbbell domain  $\Omega_\epsilon$  with  $f$  satisfying the following conditions:

- (i)  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a  $\mathcal{C}^2$  function,
- (ii)  $|f(u)| + |f'(u)| + |f''(u)| \leq C_1$  for all  $u \in \mathbb{R}$ .

**Remark 2.2.** *The condition (ii) on the nonlinearity does not represent any restriction. Since the nonlinearity is assumed dissipative, we have  $L^\infty$  estimates of the attractors of the system and these estimates are uniform in the parameter  $\epsilon$ . In particular, all solutions of*

(2.3) are bounded with a bound independent of  $\epsilon$ . In case (ii) is not satisfied we can cut off the nonlinearity without modifying the solutions of the equation so that (ii) is satisfied.

We will denote by  $\{\mathcal{E}_\epsilon\}_{\epsilon \in (0,1]}$  the set of solutions of the problem (2.3). Under the above assumptions on the nonlinearity  $f$ ,  $\mathcal{E}_\epsilon$  are bounded in  $L^\infty(\Omega_\epsilon)$  uniformly for  $\epsilon \in (0,1]$ .

The limit problem of (2.3) as  $\epsilon \rightarrow 0$  is the following

$$\begin{cases} -\Delta w + w = f(w), & x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega \\ -\frac{1}{g}(gv_s)_s + v = f(v), & s \in (0,1) \\ v(0) = w(0), & v(1) = w(1). \end{cases} \quad (2.4)$$

Observe that a solution of the limit equation has two components,  $(w, v)$ . The first one lives in  $\Omega$  and the second one lives in  $(0,1)$  or equivalently in the segment  $R_0$ . Moreover, the problem is not decoupled but it is interesting to note that it is coupled only in one direction. By this we mean that the function  $w$  is independent of  $v$  but  $v$  depends on  $w$ . Hence, to solve (2.4) we first find a solution  $w$  of the nonlinear problem in  $\Omega$ ,

$$\begin{cases} -\Delta w + w = f(w), & x \in \Omega \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \end{cases} \quad (2.5)$$

Any solution of (2.5) is very smooth. In particular, it is in  $C^0(\bar{\Omega})$  and it makes sense to take the trace of  $w$  at  $p_0$  and  $p_1$ . Once this is obtained we solve the nonlinear problem in  $(0,1)$  given by

$$\begin{cases} -\frac{1}{g}(gv_s)_s + v = f(v), & s \in (0,1) \\ v(0) = w(0), & v(1) = w(1). \end{cases} \quad (2.6)$$

The aim of this paper is to compare the solutions of the perturbed problem (2.3) and the solutions of the limit problem (2.4). Since the solutions live in different spaces we need to devise a mechanism to compare functions defined in the limiting domain  $\Omega \cup R_0$  and in  $\Omega_\epsilon$ . First of all, we need an extension operator that maps functions  $(w, v)$  defined in  $\Omega \cup R_0$  into functions defined in  $\Omega_\epsilon$ . The natural way to define this operator is to extend the functions defined in  $R_0$  (that depend only on the variable  $s$ ) constantly in the other  $N-1$  variables, that is:

$$E_\epsilon(w, v)(x) = \begin{cases} w(x), & x \in \Omega \\ v(s), & x = (s, y) \in R_\epsilon \end{cases}.$$

If we consider now  $X_\epsilon$ ,  $0 \leq \epsilon \leq 1$ , a family of functional spaces in  $\Omega_\epsilon$ , (say for instance  $X_\epsilon = L^2(\Omega_\epsilon)$ ,  $0 < \epsilon \leq 1$  and  $X_0 = L^2(\Omega) \times L^2(R_0)$ ), we can give the following definition of convergence:  $u_\epsilon \rightarrow u_0$  if  $\|u_\epsilon - E_\epsilon u_0\|_{X_\epsilon} \rightarrow 0$ . This notion of convergence will strongly depend not only on the space  $X_\epsilon$  but also, in a crucial way, on the metric chosen in  $X_\epsilon$ . For instance, if we choose  $X_\epsilon = L^2(\Omega_\epsilon)$  with the usual metric  $\|u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 = \int_{\Omega_\epsilon} |u_\epsilon|^2$ , we will have that the family of functions  $u_\epsilon \equiv 1$  will converge to any function  $u_0 \in L^2(\Omega) \times L^2(R_0)$  such



that  $u_0 = 1$  in  $\Omega$  and it is arbitrary in  $R_0$ . In particular, with this choice of metric in  $L^2(\Omega_\epsilon)$  the limit is not unique. On the other hand, if we define the metric in  $L^2(\Omega_\epsilon)$  by

$$\|u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 = \int_{\Omega} |u_\epsilon|^2 + \frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} |u_\epsilon|^2$$

we are magnifying the functions in the channel  $R_\epsilon$  with a factor  $\epsilon^{-(N-1)}$ . It is not difficult to show that with this definition, we have that the limit is unique. In particular, the functions  $u_\epsilon \equiv 1$  converge to the function  $u_0 \equiv 1$  in  $\Omega \cup R_0$ .

This considerations motivate the definition of the following spaces: for  $1 \leq p < \infty$ , the space  $U_\epsilon^p$  is the space  $L^p(\Omega_\epsilon)$  with the norm

$$\|\cdot\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|\cdot\|_{L^p(R_\epsilon)}$$

and denote by  $U_0^p = L^p(\Omega) \oplus L_g^p(0, 1)$  where  $L_g^p(0, 1)$  is the space  $L^p(0, 1)$  with the norm

$$\|u\|_{L_g^p(0,1)} = \left( \int_0^1 g(s) |u(s)|^p ds \right)^{1/p}$$

As a matter of fact, with the norm defined in  $\Omega_\epsilon$  we capture the behavior of the functions in the channel  $R_\epsilon$ . Note that a function  $u$  defined in  $\Omega_\epsilon$  but independent of the  $y$  coordinate in the channel  $R_\epsilon$  will satisfy

$$\|u\|_{U_\epsilon^p} = \|u\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|u\|_{L^p(R_\epsilon)} = \|u\|_{L^p(\Omega)} + \left( \int_0^1 g(s) u(s)^p ds \right)^{\frac{1}{p}}.$$

Notice that the extension operator  $E_\epsilon$  maps  $U_0^p$  to  $U_\epsilon^p$ . Moreover,

$$\|E_\epsilon(w, v)\|_{U_\epsilon^p} = \|(w, v)\|_{U_0^p}$$

We will also consider the spaces  $H_\epsilon^1 = H^1(\Omega) \oplus H^1(R_\epsilon)$  with the norm

$$\|\cdot\|_{H_\epsilon^1} = \|\cdot\|_{H^1(\Omega)} + \epsilon^{\frac{1-N}{2}} \|\cdot\|_{H^1(R_\epsilon)}$$

With all this notation in mind we can state one of the main result in this paper

**Theorem 2.3.** *Let  $p > N$ . If we denote by  $\mathcal{E}_\epsilon$  the set of solutions of problem (2.3) for  $\epsilon \in (0, 1]$  and by  $\mathcal{E}_0$  the set of solutions of problem (2.4) then we have the following:*

i) *For any sequence  $u_\epsilon^* \in \mathcal{E}_\epsilon$  with  $\epsilon \rightarrow 0$ , there exists a subsequence, that we also denote by  $\epsilon$ , and a  $u_0^* \in \mathcal{E}_0$  such that*

$$\|u_\epsilon^* - E_\epsilon(u_0^*)\|_{U_\epsilon^p} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0 \quad (2.7)$$

$$\|u_\epsilon^* - E_\epsilon(u_0^*)\|_{H_\epsilon^1} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0 \quad (2.8)$$

Moreover, there exists an  $\alpha \in (0, 1)$  such that for any compact set  $K \subset \bar{\Omega} \setminus \{p_0, p_1\}$  we have  $\|u_\epsilon^* - u_0^*\|_{C^{1,\alpha}(K)} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

ii) *For any hyperbolic equilibrium point  $u_0^* \in \mathcal{E}_0$ , there exists  $\eta > 0$  and  $\epsilon_0 > 0$  such that there exists one and only one equilibrium  $u_\epsilon^*$  of (2.3) such that*

$$\|u_\epsilon^* - E_\epsilon(u_0^*)\|_{U_\epsilon^p} \leq \eta \quad \text{for } 0 < \epsilon \leq \epsilon_0. \quad (2.9)$$

Moreover, (2.7) and (2.8) are satisfied.

In particular, if every equilibrium of the limit problem is hyperbolic, then we have only a finite number of them, that is,  $\mathcal{E}_0 = \{u_0^1, \dots, u_0^m\}$  and there exists an  $\epsilon_0 > 0$  such that  $\mathcal{E}_\epsilon = \{u_\epsilon^1, \dots, u_\epsilon^m\}$  and  $u_\epsilon^i \rightarrow u_0^i$  in the sense of (2.7) and (2.8). Moreover, the number of equilibria,  $m$ , is an odd number.

**Remark 2.4.** An equilibrium point  $u_0^* = (w_0, v_0) \in \mathcal{E}_0$  is hyperbolic if the linearization of (2.4) does not have any eigenvalue in the imaginary axis. Observe that  $\lambda$  is an eigenvalue of the linearization if we have solutions  $(\phi, \psi)$  not identically zero, such that

$$\begin{cases} -\Delta\phi + \phi - f'(w_0)\phi = \lambda\phi, & x \in \Omega, \\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega \\ -\frac{1}{g}(g\psi_s)_s + \psi - f'(v_0)\psi = \lambda\psi, & s \in (0, 1) \\ \psi(0) = \phi(0), \quad \psi(1) = \phi(1). \end{cases} \quad (2.10)$$

From (2.10) it is easy to see that all eigenvalues are real (although the operator obtained through linearization is not selfadjoint) and that  $\lambda = 0$  is not an eigenvalue of (2.10) if  $\lambda = 0$  is not an eigenvalue of the operator  $-\Delta\phi + \phi - f'(w_0)\phi = \lambda\phi$  in  $\Omega$  with homogeneous Neumann boundary condition nor an eigenvalue of the operator  $-\frac{1}{g}(g\psi_s)_s + \psi - f'(v_0)\psi = \lambda\psi$  in  $(0, 1)$  with homogeneous Dirichlet boundary conditions.

As a matter of fact we will be able to obtain more information on the relation between the linearized operators around equilibria. We will show the following

**Theorem 2.5.** In the conditions of Theorem 2.3 let  $u_\epsilon^*$  be a sequence of equilibria of (2.3) and  $u_0^* = (w_0, v_0)$  an equilibrium of (2.4) satisfying (2.7) and (2.8). Denote by  $\{\lambda_n^\epsilon\}_{n=1}^\infty$  the set of eigenvalues (ordered and counting multiplicity) of the linearization around  $u_\epsilon^*$ , that is, the eigenvalues of

$$\begin{cases} -\Delta\phi_\epsilon + \phi_\epsilon - f'(u_\epsilon^*)\phi_\epsilon = \lambda\phi_\epsilon, & x \in \Omega_\epsilon, \\ \frac{\partial\phi_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon, \end{cases} \quad (2.11)$$

and by  $\{\phi_n^\epsilon\}$  a corresponding set of orthonormal eigenfunctions.

Also, denote by  $\{\lambda_n^0\}_{n=1}^\infty$  the set of eigenvalues of (2.10), ordered and counting its algebraic multiplicity, and denote by  $\{\phi_n^0\}_{n=1}^\infty$  a corresponding set of generalized eigenfunctions. Then, we have

$$\lambda_n^\epsilon \xrightarrow{\epsilon \rightarrow 0} \lambda_n^0, \text{ for all } n = 1, 2, \dots$$

Also if  $n$  is such that  $\lambda_n^0 < \lambda_{n+1}^0$  and we define

$$W_n^0 = \text{span}[\phi_1^0, \dots, \phi_n^0], \quad W_n^\epsilon = \text{span}[\phi_1^\epsilon, \dots, \phi_n^\epsilon]$$

then,

$$\text{dist}_{U_\epsilon^p}(W_n^\epsilon, E_\epsilon W_n^0) \xrightarrow{\epsilon \rightarrow 0} 0, \quad \text{dist}_{H_\epsilon^1}(W_n^\epsilon, E_\epsilon W_n^0) \xrightarrow{\epsilon \rightarrow 0} 0 \quad (2.12)$$

In particular, if  $u_0^*$  is a hyperbolic equilibrium point of the limit equation and  $u_\epsilon^*$  is the sequence of equilibrium points such that  $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon^p} \xrightarrow{\epsilon \rightarrow 0} 0$  given by Theorem 2.3, then

for  $\epsilon$  small enough,  $u_\epsilon^*$  is also hyperbolic and its linearized unstable manifold converge, in the sense of (2.12), to the linearized unstable manifold of  $u_0^*$ . In particular, the dimension of the unstable manifolds of  $u_\epsilon^*$  and of  $u_0^*$  coincide.

**Remark 2.6.** *i) In relation to (2.12), the distance of two subspaces is the symmetric Hausdorff distance of the unit balls of the two subspaces, that is if  $W_1, W_2$  are subspace of the Banach space  $U$ , then*

$$\text{dist}_U(W_1, W_2) = \sup_{x \in B_{W_1}} \inf_{y \in B_{W_2}} \|x - y\|_U + \sup_{y \in B_{W_2}} \inf_{x \in B_{W_1}} \|x - y\|_U$$

where  $B_{W_1}$  and  $B_{W_2}$  are the unit balls of  $W_1$  and  $W_2$  respectively.

*ii) The convergence of the linearized unstable manifold is a first step needed to prove the convergence of the attractors. As it is mentioned in the introduction, this result will be accomplished in [7].*

The results of the above theorems will be obtained after a careful analysis on the behavior of the resolvent of the linear operators is performed. Actually, we will prove the following basic and important result:

**Proposition 2.7.** *For  $f_\epsilon \in U_\epsilon^p$ ,  $\epsilon \in (0, 1]$  let  $u_\epsilon$  be the solution of*

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon, & x \in \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon. \end{cases} \quad (2.13)$$

and for  $(f, h) \in U_0^p$  let  $(w, v)$  be the solution of

$$\begin{cases} -\Delta w + w = f, & x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega \\ -\frac{1}{g}(gv_s)_s + v = h, & s \in (0, 1) \\ v(0) = w(0), & v(1) = w(1). \end{cases} \quad (2.14)$$

Then, we have that

(1) *With  $p > N/2$ , there exists a constant  $C > 0$ , independent of  $\epsilon$  and of  $f_\epsilon$ , such that*

$$\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C \|f_\epsilon\|_{U_\epsilon^p}$$

(2) *With  $p > N$ , if  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ ,  $\epsilon \in (0, 1]$ , there is a subsequence, denoted by  $\epsilon$  again and  $(f, h) \in U_0^p$ , such that if  $(w, v)$  are given by (2.14) then the following holds*

$$\begin{aligned} i) \quad & \|u_\epsilon - E_\epsilon(w, v)\|_{H_\epsilon^1} \xrightarrow{\epsilon \rightarrow 0} 0, \\ ii) \quad & \|u_\epsilon - E_\epsilon(w, v)\|_{U_\epsilon^q} \xrightarrow{\epsilon \rightarrow 0} 0, \quad \text{for all } 1 \leq q < \infty \\ iii) \quad & \|u_\epsilon - w\|_{C^{1,\alpha}(K)} \xrightarrow{\epsilon \rightarrow 0} 0, \quad \text{for all compact } K \subset \bar{\Omega} \setminus p_0, p_1 \end{aligned}$$

(3) *With  $p > N$ , if we have  $\|f_\epsilon - E_\epsilon(f, h)\|_{U_\epsilon^p} \xrightarrow{\epsilon \rightarrow 0} 0$  then we have i), ii) and iii) for the whole sequence*

The proof of this Proposition is written in Appendix A.

### 3. THE PROBLEMS (2.3) AND (2.4)

We will write both problems, (2.3) and (2.4) as abstract problems in the Banach spaces  $U_\epsilon^p$  and  $U_p^0$  respectively.

Since for fixed  $\epsilon$ , the space  $U_\epsilon^p$  is equivalent to  $L^p(\Omega_\epsilon)$ , the problem (2.3) can be written as an abstract equation of semilinear type of the form

$$A_\epsilon u = F_\epsilon(u), \quad (3.1)$$

where  $A_\epsilon : \mathcal{D}(A_\epsilon) \subset U_\epsilon^p \rightarrow U_\epsilon^p$ ,  $1 \leq p < \infty$ , is the linear operator defined by

$$\begin{aligned} \mathcal{D}(A_\epsilon) &= \{u \in W^{2,p}(\Omega_\epsilon) : \Delta u \in U_\epsilon^p, \partial u / \partial n = 0 \text{ in } \partial\Omega_\epsilon\} \\ A_\epsilon u &= -\Delta u + u, \quad u \in \mathcal{D}(A_\epsilon) \end{aligned} \quad (3.2)$$

and the nonlinearity  $F_\epsilon : U_\epsilon \rightarrow U_\epsilon$  is the Nemitskii operator generated by  $f$ , that is  $F_\epsilon(u_\epsilon)(x) = f(u_\epsilon(x))$ .

The operator  $A_\epsilon$  is sectorial and the following estimate holds

$$\|(A_\epsilon + \lambda)^{-1}\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \leq \frac{C}{|\lambda|}, \quad \text{for } \lambda \in \Sigma_\theta, \quad (3.3)$$

where  $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda - 1)| \leq \theta\}$ ,  $\theta > \frac{\pi}{2}$  and  $C$  is a constant that does not depend on  $\epsilon$ . This follows from the fact that the localization of the numerical range in the complex plane can be done independently of  $\epsilon$ , see [37], page 215.

Define the limit linear operator,  $A_0 : \mathcal{D}(A_0) \subset U_0^p \rightarrow U_0^p$  which is defined by

$$A_0(w, v) = \left( -\Delta w + w, -\frac{1}{g}(gv_x)_x + v \right), \quad (w, v) \in \mathcal{D}(A_0) \quad (3.4)$$

with domain

$$\begin{aligned} \mathcal{D}(A_0) &= \{(w, v) \in U_0^p : w \in \mathcal{D}(\Delta_N^\Omega), (gv_x)_x \in L_g^p(0, 1), \\ &\quad v(0) = w(0), v(1) = w(1)\} \end{aligned} \quad (3.5)$$

where  $\Delta_N^\Omega$  is the Laplace operator with homogeneous Neumann boundary conditions in  $L^p(\Omega)$ .

We have the following

**Proposition 3.1.** *The operator  $A_0$  defined by (3.4) has the following properties*

- (i)  $\mathcal{D}(A_0)$  is dense in  $U_0^p$ ,
- (ii) If  $p > N/2$  then  $A_0$  is a closed operator,
- (iii)  $A_0$  has compact resolvent.

**Proof:** (i) Let  $(w, v) \in L^p(\Omega) \oplus L_g^p(0, 1)$ . Let  $(w_n, v_n) \in C_0^\infty(\Omega) \oplus C_0^\infty(0, 1)$  with  $(w_n, v_n) \rightarrow (w, v)$  in  $L^p(\Omega) \oplus L_g^p(0, 1)$ , then  $(w_n, v_n) \in \mathcal{D}(A_0)$  and the result is proved.

(ii) Let  $(w_n, v_n) \in \mathcal{D}(A_0)$  be such that  $(w_n, v_n) \rightarrow (w, v)$  and  $A_0(w_n, v_n) \rightarrow (\phi, \psi)$  in  $L^p(\Omega) \oplus L_g^p(0, 1)$ . Since  $w_n \in \mathcal{D}(\Delta_N^\Omega)$  and  $\Delta_N^\Omega$  is a closed operator in  $L^p(\Omega)$ , see [20], we have that  $w \in \mathcal{D}(\Delta_N^\Omega)$  and  $w_n \rightarrow w$  in  $W^{2,p}(\Omega)$ . In particular,  $-\Delta w_n \rightarrow -\Delta w$  and since  $p > N/2$  we

have  $W^{2,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ , which implies that  $w_n(0) \rightarrow w(0)$  and  $w_n(1) \rightarrow w(1)$ . On the other hand  $v_n \rightarrow v$  and  $\psi_n = -\frac{1}{g}(g v_n')' + v_n \rightarrow \psi$  in  $L_g^p(0, 1)$ . Now

$$\begin{cases} -\frac{1}{g}(g v_n')' + v_n = \psi_n, & s \in (0, 1) \\ v_n(0) = w_n(0), & v_n(1) = w_n(1). \end{cases}$$

Making the change of variables  $z_n = v_n - \xi_n$ , where  $\xi_n$  is the solution of the following problem

$$\begin{cases} -\frac{1}{g}(g \xi_n')' = 0, & s \in (0, 1) \\ \xi_n(0) = w_n(0), & \xi_n(1) = w_n(1), \end{cases} \quad (3.6)$$

we have

$$\begin{cases} -\frac{1}{g}(g z_n')' = \psi_n - v_n, & s \in (0, 1) \\ z_n(0) = z_n(1) = 0. \end{cases}$$

It is easy to see that

$$\xi_n(s) = w_n(0) + \frac{w_n(1) - w_n(0)}{\int_0^1 \frac{1}{g(\theta)} d\theta} \int_0^s \frac{1}{g(\theta)} d\theta \quad (3.7)$$

and, since  $w_n(0) \rightarrow w(0)$ ,  $w_n(1) \rightarrow w(1)$ , it follows that  $\xi_n \rightarrow \xi$ , where  $\xi$  is the solution of the following problem

$$\begin{cases} -\frac{1}{g}(g \xi')' = 0, & s \in (0, 1) \\ \xi(0) = w(0), & \xi(1) = w(1). \end{cases} \quad (3.8)$$

Moreover, since the operator  $L(v) = -\frac{1}{g}(g v')'$  with homogeneous Dirichlet boundary conditions at  $s = 0$  and  $s = 1$  is closed in  $L_g^p(0, 1)$ , we have that  $z_n \rightarrow z$  in  $L_g^p(0, 1)$  where  $z$  satisfies

$$\begin{cases} -\frac{1}{g}(g z')' = \psi - v, & s \in (0, 1) \\ z(0) = z(1) = 0. \end{cases}$$

From which it follows that  $v_n = z_n + \xi_n \rightarrow z + \xi = v$ , and  $v$  satisfies

$$\begin{cases} -\frac{1}{g}(g v')' + v = \psi, & s \in (0, 1) \\ v(0) = w(0), & v(1) = w(1). \end{cases} \quad (3.9)$$

(iii) Since  $\mathcal{D}(A_0) \subset W^{2,p}(\Omega) \oplus W_g^{1,p}(0, 1) \hookrightarrow L^p(\Omega) \oplus L_g^p(0, 1)$  and since the embedding  $W^{2,p}(\Omega) \oplus W_g^{1,p}(0, 1) \hookrightarrow L^p(\Omega) \oplus L_g^p(0, 1)$  is compact, it follows that  $A_0$  has compact resolvent. ■

**Remark 3.2.** *Even though Proposition 3.1 states several important properties of the operator  $A_0$ , we would like to mention that  $A_0$  is not a sectorial operator. Its spectrum is all real and, therefore, it is contained in a sector but the required resolvent estimate*

$$\|(A_0 + \lambda I)^{-1}\|_{\mathcal{L}(U_0^p)} \leq \frac{C}{|\lambda + a|}$$

*is not satisfied. To see this, we refer to [7].*

#### 4. ABSTRACT COMPACT CONVERGENCE RESULTS

In this section we develop the basic abstract tool that we are going to use to compare two linear problems defined in different spaces. This theory will be applied to compare the linear problem defined in the dumbbell domain  $\Omega_\epsilon$  with the linear problem defined in the limit domain  $\Omega \cup R_0$ . This will be illustrated throughout several examples included in the section.

Hence, let  $U_\epsilon$  be a family of Banach spaces for  $\epsilon \in [0, 1]$  and assume there is a family of linear operators  $E_\epsilon : U_0 \rightarrow U_\epsilon$  with the property that

$$\|E_\epsilon u\|_{U_\epsilon} \xrightarrow{\epsilon \rightarrow 0} \|u\|_{U_0}, \quad \text{for all } u \in U_0. \quad (4.1)$$

**Example 4.1.** *Let  $\Omega_\epsilon = \Omega \cup R_\epsilon$  be the dumbbell domain defined in Section 2 and let  $U_\epsilon^p$  and  $U_0^p$  be the spaces defined also in Section 2. Consider the extension operators  $E_\epsilon : U_0^p \rightarrow U_\epsilon^p$  as*

$$E_\epsilon(w, v)(x) = \begin{cases} w(x), & x \in \Omega \\ v(s), & x = (s, y) \in R_\epsilon \end{cases}.$$

*It is very easy to verify that  $\|E_\epsilon(w, v)\|_{U_\epsilon^p} = \|(w, v)\|_{U_0^p}$ .*

**Definition 4.2.** *We say that a sequence  $\{u_\epsilon\}_{\epsilon \in (0, 1]}$   $E$ -converges to  $u$  if  $\|u_\epsilon - E_\epsilon u\|_{U_\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0$ . We write this as  $u_\epsilon \xrightarrow{E} u$ .*

With this notion of convergence we introduce the notion of compactness

**Definition 4.3.** *A sequence  $\{u_n\}_{n \in \mathbb{N}}$ , with  $u_n \in U_{\epsilon_n}$  and  $\epsilon_n \rightarrow 0$ , is said pre-compact if for each subsequence  $\{u_{n'}\}$  there is another subsequence  $\{u_{n''}\}$  and an element  $u \in U_0$  such that  $u_{\epsilon_{n''}} \xrightarrow{E} u$ . The family  $\{u_\epsilon\}_{\epsilon \in (0, 1]}$  is said pre-compact if for each sequence  $\{u_{\epsilon_n}\}$ , with  $\epsilon_n \rightarrow 0$ , is pre-compact.*

**Definition 4.4.** *We say that a family of operators  $\{B_\epsilon \in L(U_\epsilon) : \epsilon \in (0, 1]\}$  converges to  $B_0 \in L(U_0)$ , as  $\epsilon \rightarrow 0$ , if  $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$  whenever  $u_\epsilon \xrightarrow{E} u \in U_0$ . We write  $B_\epsilon \xrightarrow{EE} B_0$ .*

**Definition 4.5.** *We say that a family of compact operators  $\{B_\epsilon \in L(U_\epsilon) : \epsilon \in (0, 1]\}$  converges compactly to a compact operator  $B_0 \in L(U_0)$  if for any family  $\{u_\epsilon\}$  with  $\|u_\epsilon\|_{U_\epsilon} = 1$ ,  $\epsilon \in (0, 1]$ , the family  $\{B_\epsilon u_\epsilon\}$  is relatively compact and, moreover,  $B_\epsilon \xrightarrow{EE} B_0$ . We denote this as  $B_\epsilon \xrightarrow{CC} B_0$ .*

**Example 4.6.** Let  $\Omega_\epsilon, \Omega_0, U_\epsilon^p, U_0^p$  be the domains and the spaces of the dumbbell domain of Example 4.1, let  $A_\epsilon, A_0$  the operators defined in Section 3 and consider the operators  $B_\epsilon \in L(U_\epsilon^p)$  defined by  $B_\epsilon = A_\epsilon^{-1}$ , that is,  $B_\epsilon f_\epsilon = u_\epsilon$  where  $u_\epsilon$  is the solution of

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f_\epsilon, & x \in \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} = 0, & x \in \partial\Omega_\epsilon. \end{cases} \quad (4.2)$$

and  $B_0 \in L(U_0^p)$  be the operator defined by  $B_0 = A_0^{-1}$ , that is  $B_0(f, h) = (u, v)$  where  $(u, v)$  is the solution of

$$\begin{cases} -\Delta w + w = f, & x \in \Omega \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ -\frac{1}{g(s)}(g(s)v'(s))' + v(s) = h(s), & s \in (0, 1) \\ v(0) = w(0), & v(1) = w(1). \end{cases} \quad (4.3)$$

We will prove in Appendix A that if  $p > N$ , then  $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$ . This is the fundamental result that will give us the key to all the results of the paper. Also, notice that this is exactly what Proposition 2.7 states.

The following lemma is a key result.

**Lemma 4.7.** Assume that  $\{B_\epsilon \in L(U_\epsilon)\}_{\epsilon \in (0,1]}$  converges compactly to  $B_0$  as  $\epsilon \rightarrow 0$ . Then,  
i)  $\|B_\epsilon\|_{\mathcal{L}(U_\epsilon)} \leq C$  for some constant  $C$ , independent of  $\epsilon$ .  
ii) Assume that  $\mathcal{N}(I + B_0) = \{0\}$  then, there exists an  $\epsilon_0 > 0$  and  $M > 0$  such that

$$\|(I + B_\epsilon)^{-1}\|_{L(U_\epsilon)} \leq M, \quad \forall \epsilon \in [0, \epsilon_0]. \quad (4.4)$$

**Proof:** i) If the norms are not bounded, then we can choose a sequence of  $\epsilon_n \rightarrow 0$  and  $u_{\epsilon_n} \in U_{\epsilon_n}$  with  $\|u_{\epsilon_n}\|_{U_{\epsilon_n}} = 1$  such that  $\|B_{\epsilon_n} u_{\epsilon_n}\| \rightarrow +\infty$ . But this is in contradiction with the compact convergence of  $B_\epsilon$  given in Definition 4.5.

ii) Because  $B_\epsilon$  is compact for every  $\epsilon \in [0, 1]$ , the estimate (4.4) is equivalent to say that

$$\|(I + B_\epsilon)u_\epsilon\|_{U_\epsilon} \geq \frac{1}{M}, \quad \forall \epsilon \in [0, \epsilon_0] \text{ and } \forall u_\epsilon \in U_\epsilon \text{ with } \|u_\epsilon\| = 1.$$

Suppose that this is not true; that is, suppose that there is a sequence  $\{u_n\}$ , with  $u_n \in U_{\epsilon_n}$ ,  $\|u_n\| = 1$  and  $\epsilon_n \rightarrow 0$  such that  $\|(I + B_{\epsilon_n})u_n\| \rightarrow 0$ . Since  $\{B_{\epsilon_n} u_n\}$  has a convergent subsequence, which we again denote by  $\{B_{\epsilon_n} u_n\}$ , to  $u$ ,  $\|u\| = 1$ , then  $u_n + B_{\epsilon_n} u_n \rightarrow 0$  and  $u_n \rightarrow -u$ . This implies that  $(I + B)u = 0$  contradicting our hypothesis.  $\blacksquare$

In general, we will have that the operators  $B_\epsilon$  are inverses of certain differential operators  $A_\epsilon$ . Hence, assume we have operators  $A_\epsilon : \mathcal{D}(A_\epsilon) : U_\epsilon \rightarrow U_\epsilon$  for  $\epsilon \in [0, 1]$  and assume that we have the following hypotheses:

$A_\epsilon \text{ is closed, has compact resolvent, } 0 \in \rho(A_\epsilon), \epsilon \in [0, 1] \text{ and } A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}.$

(4.5)

**Lemma 4.8.** *Let  $A_\epsilon$  be such that (4.5) hold. Then, for any  $\lambda \in \rho(A_0)$ , there is an  $\epsilon_\lambda > 0$  such that  $\lambda \in \rho(A_\epsilon)$  for all  $\epsilon \in [0, \epsilon_\lambda]$  and there is a constant  $M_\lambda > 0$  such that*

$$\|(\lambda - A_\epsilon)^{-1}\| \leq M_\lambda, \quad \forall \epsilon \in [0, \epsilon_\lambda]. \quad (4.6)$$

Furthermore,  $(\lambda - A_\epsilon)^{-1}$  converges compactly to  $(\lambda - A_0)^{-1}$  as  $\epsilon \rightarrow 0$ .

**Proof:** From (4.5) and since  $\lambda \in \rho(A_0)$  it is easy to see that

$$(\lambda - A_0)^{-1} = -A_0^{-1}(I - \lambda A_0^{-1})^{-1}.$$

Since  $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$ , applying Lemma 4.7 i) and ii), we get that the operator  $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1}$  is well defined and bounded. Easy computations show that actually  $-A_\epsilon^{-1}(I - \lambda A_\epsilon^{-1})^{-1} = (\lambda - A_\epsilon)^{-1}$ . Hence  $\lambda \in \rho(A_\epsilon)$  and we obtain (4.6).

In order to show the compact convergence of  $(\lambda - A_\epsilon)^{-1}$  to  $(\lambda - A_0)^{-1}$  we proceed as follows.

Since  $A_\epsilon^{-1}$  converges compactly to  $A_0^{-1}$  and since  $\{(I - \lambda A_\epsilon^{-1}) : 0 \leq \epsilon \leq \epsilon_\lambda\}$  is bounded we conclude that

- If  $\|u_\epsilon\|_{U_\epsilon} = 1$  then  $(\lambda - A_\epsilon)^{-1}u_\epsilon = -A_\epsilon^{-1}w_\epsilon$  with  $w_\epsilon = (I - \lambda A_\epsilon^{-1})^{-1}u_\epsilon$  which is uniformly bounded in  $\epsilon$ . Hence  $(\lambda - A_\epsilon)^{-1}u_\epsilon$  has an  $E$ -convergent subsequence.
- If  $u_\epsilon \xrightarrow{E} u$  then  $A_\epsilon^{-1}u_\epsilon \xrightarrow{E} A_0^{-1}u$ . Now, for any subsequence of  $\{(\lambda - A_\epsilon)^{-1}u_\epsilon\}$  there is a subsequence (which we again denote by  $\{(\lambda - A_\epsilon)^{-1}u_\epsilon\}$ ) and a  $y$  such that,

$$(\lambda - A_\epsilon)^{-1}u_\epsilon = -(I - \lambda A_\epsilon^{-1})^{-1}A_\epsilon^{-1}u_\epsilon = z_\epsilon \xrightarrow{E} y.$$

Therefore,

$$A_0^{-1}u \xleftarrow{E} A_\epsilon^{-1}u_\epsilon = -(I - \lambda A_\epsilon^{-1})z_\epsilon \xrightarrow{E} -(I - \lambda A_0^{-1})y$$

This implies that  $y = (\lambda - A_0)^{-1}u$ . In particular,  $y$  is independent of the subsequence chosen. This implies that the whole sequence  $(\lambda - A_\epsilon)^{-1}u_\epsilon$  converges to  $y = (\lambda - A_0)^{-1}u$ . Thus,  $(\lambda - A_\epsilon)^{-1} \xrightarrow{EE} (\lambda - A_0)^{-1}$ .

From this we have the compact convergence of  $(\lambda - A_\epsilon)^{-1} \xrightarrow{CC} (\lambda - A_0)^{-1}$  and the result is proved.  $\blacksquare$

**Lemma 4.9.** *If  $\lambda$  and  $\delta$  are chosen such that  $S_\delta := \{\mu \in \mathbb{C} : |\mu - \lambda| = \delta\}$  satisfies  $\sigma(A_0) \cap S_\delta = \emptyset$  then, there exists  $\epsilon_{S_\delta} > 0$  such that  $\sigma(A_\epsilon) \cap S_\delta = \emptyset$  for all  $\epsilon \leq \epsilon_{S_\delta}$ .*

**Proof:** Suppose not. Then, there are sequences  $\epsilon_n \rightarrow 0$ ,  $\lambda_n \in S_\delta$  (which we may assume convergent to  $\lambda$ ) and  $u_{\epsilon_n} \in U_{\epsilon_n}$ ,  $\|u_{\epsilon_n}\| = 1$  such that  $u_{\epsilon_n} - (A_{\epsilon_n})^{-1}\lambda_n u_{\epsilon_n} = 0$  or equivalently  $\lambda_n(A_{\epsilon_n})^{-1}u_{\epsilon_n} = u_{\epsilon_n}$ . It follows from compact convergence that  $u_{\epsilon_n}$  has a convergent subsequence to  $u \in U_0$ ,  $\|u\|_{U_0} = 1$  and that  $A_0 u = \lambda u$  which contradicts our assumption.  $\blacksquare$

For an isolated point  $\lambda \in \sigma(A_0)$  we associate its generalized eigenspace  $W(\lambda, A_0) = Q(\lambda, A_0)U_0$  where

$$Q(\lambda, A_0) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi I - A_0)^{-1} d\xi$$



and  $\delta$  is chosen so small that there is no other point of  $\sigma(A_0)$  in the disc  $\{\xi \in \mathbb{C} : |\xi - \lambda| \leq \delta\}$ . It follows from the previous Lemma that there is  $\epsilon_{S_\delta}$  such that  $\rho(A_\epsilon) \supset S_\delta$  for all  $\epsilon \leq \epsilon_{S_\delta}$ . We denote by  $W(\lambda, A_\epsilon) = Q(\lambda, A_\epsilon)U_\epsilon$  where

$$Q(\lambda, A_\epsilon) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi I - A_\epsilon)^{-1} d\xi.$$

Our next result says that the spectrum of  $A_\epsilon$ , for  $\epsilon$  small, *approaches* the spectrum of  $A_0$ . We already know that the spectrum of  $A_\epsilon$  or  $A_0$  consists of isolated eigenvalues only.

**Theorem 4.10.** *Let  $A_\epsilon, A_0$  be such that (4.5) is satisfied. Then the following statements hold.*

- (i) *If  $\lambda_0 \in \sigma(A_0)$ , there exists a sequence  $\epsilon_n \rightarrow 0$  and  $\lambda_n \in \sigma(A_{\epsilon_n})$ ,  $n \in \mathbb{N}$ , such that  $\lambda_n \rightarrow \lambda_0$  as  $n \rightarrow \infty$ .*
- (ii) *If for some sequences  $\epsilon_n \rightarrow 0$ ,  $\lambda_n \in \sigma(A_{\epsilon_n})$ ,  $n \in \mathbb{N}$ , one has  $\lambda_n \rightarrow \lambda_0$  as  $n \rightarrow \infty$ , then  $\lambda_0 \in \sigma(A_0)$ .*
- (iii) *There exists  $\epsilon_0 > 0$  such that  $\dim W(\lambda, A_\epsilon) = \dim W(\lambda_0, A_0)$  for all  $0 \leq \epsilon \leq \epsilon_0$ .*
- (iv) *If  $u \in W(\lambda_0, A_0)$ , there exists a sequence  $\{u_\epsilon\}$ ,  $u_\epsilon \in W(\lambda_0, A_\epsilon)$ , such that  $u_\epsilon \xrightarrow{E} u$ .*
- (v) *If  $\epsilon_n \rightarrow 0$ , and  $u_n \in W(\lambda, A_{\epsilon_n})$ , satisfies  $\|u_n\|_{U_{\epsilon_n}} = 1$  then,  $\{u_n\}$  has an  $E$ -convergent subsequence and any limit point of this sequence belongs to  $W(\lambda_0, A_0)$ .*

**Proof.** (i) Let us take some  $\lambda_0 \in \sigma(A_0)$  and consider  $\mathcal{O}(\lambda_0, \delta) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \delta\}$  such that  $\mathcal{O}(\lambda_0, \delta) \cap \sigma(A_0) = \{\lambda_0\}$ . To show that there is  $\epsilon_0 > 0$  such that  $\|(\lambda - A_\epsilon)^{-1}\| = O(1)$  for  $\epsilon \in [0, \epsilon_0]$  and  $\lambda \in \partial\mathcal{O}(\lambda_0, \delta)$  it is enough to prove that

$$\|(I - \lambda A_\epsilon^{-1})^{-1}\| = O(1), \quad \epsilon \in [0, \epsilon_0], \quad \lambda \in \partial\mathcal{O}(\lambda_0, \delta).$$

If that is not the case there will be a sequence  $\lambda_n \in \partial\mathcal{O}(\lambda_0, \delta)$  (which we may assume convergent to some  $\tilde{\lambda} \in \partial\mathcal{O}(\lambda_0, \delta)$ ), a sequence  $u_n \in U_{\epsilon_n}$ ,  $\|u_n\| = 1$ , and a sequence  $\epsilon_n \rightarrow 0$  such that

$$\|(I - \lambda_n(A_{\epsilon_n})^{-1})u_n\| \xrightarrow{n \rightarrow \infty} 0.$$

Since  $\tilde{\lambda} \in \rho(A_0)$ , that is in contradiction with Lemma 4.7.

Assume now that  $\mathcal{O}(\lambda_0, \delta) \subset \rho(A_\epsilon)$ . The function  $(\lambda - A_\epsilon)^{-1}$  is holomorphic. From what we have just proved and from the Maximum Modulus Theorem one can see that

$$\|(I - \lambda_0 A_\epsilon^{-1})^{-1}\| \leq \max_{\substack{|\lambda - \lambda_0| = \delta \\ \epsilon \in [0, \epsilon_0]}} \|(I - \lambda A_\epsilon^{-1})^{-1}\| = c < \infty.$$

Hence if  $\epsilon_n \rightarrow 0$  and  $u_n \rightarrow u$ , it follows from Lemma 4.7 that

$$\|(\lambda_0 A_0^{-1} - I)u\|_{U_0} = \lim_{n \rightarrow \infty} \|(\lambda_0 A_{\epsilon_n}^{-1} - I)u_n\|_{U_0} \geq c\|u\|_{U_0},$$

for some  $c > 0$  and  $\lambda_0 \in \rho(A_0)$ . So any  $\mathcal{O}(\lambda_0, \delta)$  contains some point of  $\sigma(A_\epsilon)$ , for suitably small  $\epsilon$ .

(ii) Assume now that  $\epsilon_n \rightarrow 0$ ,  $\{\lambda_n\}$ ,  $\lambda_n \in \sigma(A_{\epsilon_n})$ , is such that  $\lambda_n \rightarrow \lambda$  and  $\|(I - \lambda_n(A_{\epsilon_n})^{-1})u_n\| = 0$ ,  $\|u_n\| = 1$ . Then

$$\|(I - \lambda(A_{\epsilon_n})^{-1})u_n\|_{U_{\epsilon_n}} = \|(I - \lambda_n(A_{\epsilon_n})^{-1})u_n - (\lambda - \lambda_n)(A_{\epsilon_n})^{-1}u_n\|_{U_{\epsilon_n}} \rightarrow 0$$

as  $n \rightarrow \infty$ . Once  $\|u_n\| = 1$  we have, taking subsequences if necessary,  $(A_{\epsilon_n})^{-1}u_n \rightarrow y$  and  $u_n \rightarrow u$ ,  $\|u\| = 1$ . Therefore  $u - \lambda A_0^{-1}u = 0$ ,  $u \neq 0$ , which means  $\lambda \in \sigma(A_0)$ .

(iii) Since  $(\lambda - A_\epsilon)^{-1} \xrightarrow{\epsilon \rightarrow 0} (\lambda - A_0)^{-1}$  for any  $\lambda$  such that  $|\lambda - \lambda_0| = \delta$  and since

$$\{\|(\lambda - A_\epsilon)^{-1}\| : 0 \leq \epsilon \leq \epsilon_0\}$$

is bounded, it follows from Dominated Convergence Theorem that  $Q_\epsilon(\lambda_0) \xrightarrow{\epsilon \rightarrow 0} Q(\lambda_0)$ .

If  $v_1, \dots, v_k$  is a basis for  $W(\lambda_0, A_0) = Q_0(\lambda_0)U_0$ , it is easy to see that, for suitably small  $\epsilon$ ,

$$\{Q_\epsilon(\lambda_0)E_\epsilon v_1, \dots, Q_\epsilon(\lambda_0)E_\epsilon v_k\}$$

is a linearly independent set in  $Q_\epsilon(\lambda_0)U_\epsilon$ . Hence  $\text{rank}(Q_\epsilon(\lambda_0)) \geq \text{rank}(Q(\lambda_0))$ .

We prove the converse inequality assuming that  $Q_\epsilon(\lambda_0) \rightarrow Q(\lambda_0)$  compactly. If for some sequence  $\epsilon_n \rightarrow 0$ ,  $\text{rank}(Q_{\epsilon_n}(\lambda_0)) > \text{rank}(Q(\lambda_0))$ , it follows from Lemma IV.2.3 in Kato that, for each  $n \in \mathbb{N}$  there is a  $u_n \in W(\lambda_0, A_{\epsilon_n})$ ,  $\|u_n\| = 1$ , such that  $\text{dist}(u_n, W(\lambda_0, A_0)) = 1$ . From the compact convergence we can assume that  $Q_{\epsilon_n}(\lambda_0)u_n = u_n \rightarrow Q_0(\lambda_0)u_0 = u_0$ , hence

$$1 \leq \|u_n - Q_0(\lambda_0)u_n\| = \|Q_{\epsilon_n}(\lambda_0)u_n - Q_0(\lambda_0)u_n\| \rightarrow 0$$

So we need to prove just compact convergence  $Q_\epsilon(\lambda_0) \rightarrow Q(\lambda_0)$  and that follows from the compact convergence of  $A_\epsilon^{-1} \rightarrow A_0^{-1}$ , from the uniform boundedness of  $\|(\zeta A_\epsilon^{-1} - I)^{-1}\|$  ( $|\zeta - \lambda_0| = \delta$  and  $\epsilon \in [0, \epsilon_0]$ ), given by Lemma 4.7, and from the formula

$$Q_\epsilon(\lambda_0) = \frac{1}{2\pi i} \int_{|\zeta - \lambda_0| = \delta} (\zeta I - A_\epsilon)^{-1} d\zeta = A_\epsilon^{-1} \frac{1}{2\pi i} \int_{|\zeta - \lambda_0| = \delta} (\zeta A_\epsilon^{-1} - I)^{-1} d\zeta.$$

(iv) This follows taking  $u_\epsilon = Q_\epsilon(\lambda_0)E_\epsilon u$ .

(v) Follows from the compact convergence of  $Q_\epsilon$  to  $Q$  proved in (iii). ■

**Proposition 4.11.** *Let  $A_\epsilon, A_0$  be such that condition (4.5) is satisfied. Let  $K$  be a compact subset of  $\rho(A_0)$ . Then, there is a constant  $\epsilon_K > 0$  such that  $K \subset \rho(A_\epsilon)$  for all  $\epsilon \in [0, \epsilon_K]$  and*

$$\sup_{\substack{\lambda \in K \\ \epsilon \in [0, \epsilon_K]}} \|(\lambda - A_\epsilon)^{-1}\| < \infty. \quad (4.7)$$

Furthermore, for any  $u \in U_0$

$$\sup_{\lambda \in K} \|(\lambda - A_\epsilon)^{-1}E_\epsilon u - E_\epsilon(\lambda - A_0)^{-1}u\|_{U_\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0. \quad (4.8)$$

**Proof:** Let us first prove that there is a  $\epsilon_K > 0$  such that  $K \subset \rho(A_\epsilon)$  for all  $\epsilon \in [0, \epsilon_K]$ . Suppose that this is not the case then, there are sequences  $\epsilon_n \rightarrow 0$ ,  $\{\lambda_n\} \in K$  such that  $\lambda_n$  is an eigenvalue of  $A_{\epsilon_n}$ . Since  $K$  is compact we may assume that there is a  $\bar{\lambda} \in K$  such that  $\lambda_n \rightarrow \bar{\lambda}$ . It follows from Theorem 4.10, part (ii), that  $\bar{\lambda} \in \sigma(A_0)$  which is a contradiction.

To prove (4.7), it is enough to prove that

$$\sup_{\substack{\lambda \in K \\ \epsilon \in [0, \epsilon_K]}} \|(I - \lambda A_\epsilon^{-1})^{-1}\| < \infty.$$

we assume that this is not the case; that is, assume that there are sequences  $\epsilon_n \rightarrow 0$ ,  $\lambda_n \in K$  (which we may assume convergent to  $\bar{\lambda} \in K$ ) such that

$$\|(I - \lambda_n A_{\epsilon_n}^{-1})^{-1}\| \rightarrow \infty.$$

Since  $\lambda_n A_{\epsilon_n}^{-1}$  converges compactly to  $\bar{\lambda} A_0^{-1}$  this is in contradiction with Lemma 4.7.

It remains to prove (4.8). Once again, we prove it by contradiction. Assume that there are sequences  $\epsilon_n \rightarrow 0$ ,  $K \ni \lambda_n \rightarrow \bar{\lambda} \in K$  and  $\eta > 0$  such that

$$\|(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - E_{\epsilon_n} (\lambda_n - A_0)^{-1} u\|_{U_{\epsilon_n}} \geq \eta. \quad (4.9)$$

Using the resolvent identity we have

$$(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u = (\bar{\lambda} - \lambda_n)(\lambda_n - A_{\epsilon_n})^{-1} (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u.$$

It follows from the (4.7) that

$$\|(\lambda_n - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - (\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u\|_{U_{\epsilon_n}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.10)$$

Since, from Lemma 4.8,

$$\|(\bar{\lambda} - A_{\epsilon_n})^{-1} E_{\epsilon_n} u - E_{\epsilon_n} (\bar{\lambda} - A_0)^{-1} u\|_{U_{\epsilon_n}} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (4.11)$$

and, from the continuity properties of the resolvent operators,

$$\|(\lambda_n - A_0)^{-1} u - (\bar{\lambda} - A_0)^{-1} u\|_{U_0} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.12)$$

Now, (4.10), (4.11) and (4.12) are in contradiction with (4.9) and the result is proved.  $\blacksquare$

**4.1. Linearization.** In many instances we will be interested in analyzing the behavior, in terms of compact convergence, spectrum, etc., of operators that come from the linearization around certain stationary solutions of nonlinear problems. This amounts to study the behavior of operators of the form  $A_\epsilon + V_\epsilon$  where  $V_\epsilon : U_\epsilon \rightarrow U_\epsilon$  is a bounded operator (typically a multiplication by a potential). We will see that under fairly general hypotheses, once compact convergence of  $A_\epsilon^{-1}$  to  $A_0^{-1}$  is obtained, we can analyze the operators of the form  $A_\epsilon + V_\epsilon$ .

Consider the following hypothesis

$$\boxed{(4.5) \text{ holds and } V_\epsilon \in \mathcal{L}(U_\epsilon, U_\epsilon), \epsilon \in [0, 1] \text{ such that } A_\epsilon^{-1} V_\epsilon \xrightarrow{CC} A_0^{-1} V_0.} \quad (4.13)$$

**Example 4.12.** Assume we are in the setting of Examples 4.1 and 4.6 and let  $V_\epsilon \in L^\infty(\Omega_\epsilon)$  and  $V_0 \in L^\infty(\Omega) \oplus L^\infty(0, 1)$  be potentials satisfying that  $V_\epsilon \xrightarrow{E} V_0$ . Then, we have

$$A_\epsilon^{-1} V_\epsilon \xrightarrow{CC} A_0^{-1} V_0.$$

Note that  $A_\epsilon^{-1} V_\epsilon(u_\epsilon) = A_\epsilon^{-1}(V_\epsilon u_\epsilon)$ . To prove this, notice that by the boundedness of the potentials  $V_\epsilon$  it is easy to see that if  $u_\epsilon$  is a bounded sequence in  $U_\epsilon^p$ , then  $V_\epsilon u_\epsilon$  is also a bounded sequence in  $U_\epsilon^p$ . By the compact convergence of  $A_\epsilon^{-1}$  we get that  $A_\epsilon^{-1}(V_\epsilon u_\epsilon)$  is precompact.

Moreover if  $u_\epsilon \xrightarrow{E} u_0$  in  $U_\epsilon^p$ , then  $V_\epsilon u_\epsilon \xrightarrow{E} V_0 u_0$ . And therefore  $A_\epsilon^{-1} V_\epsilon u_\epsilon \xrightarrow{E} A_0^{-1} V_0 u_0$  since  $A_\epsilon^{-1} \xrightarrow{EE} A_0^{-1}$ .

We assume the following condition

$$\boxed{0 \notin \sigma(A_0 + V_0).} \quad (4.14)$$

It is clear that  $A_0 + V_0$  has compact resolvent. Let  $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$ ,  $0 \leq \epsilon \leq 1$ . We can show the following result,

**Proposition 4.13.** *Assume that conditions (4.13) and (4.14) are satisfied. Then, there is an  $\epsilon_0 > 0$  such that  $0 \notin \sigma(A_\epsilon + V_\epsilon)$ ,  $\|(A_\epsilon + V_\epsilon)^{-1}\|_{\mathcal{L}(U_\epsilon)} \leq M$  independent of  $\epsilon$  for  $0 \leq \epsilon \leq \epsilon_0$ . Moreover*

$$(A_\epsilon + V_\epsilon)^{-1} \xrightarrow{CC} (A_0 + V_0)^{-1}.$$

*In particular, the operators  $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$ ,  $0 \leq \epsilon \leq 1$ , satisfy condition (4.5).*

**Proof:** To prove the result note that

$$(A_\epsilon + V_\epsilon)^{-1} = (I + A_\epsilon^{-1}V_\epsilon)^{-1}A_\epsilon^{-1}$$

Since  $-A_\epsilon^{-1}V_\epsilon$  converges compactly to  $-A_0^{-1}V_0$  and  $-A_\epsilon^{-1}$  converges compactly to  $(-A_0)^{-1}$ , the uniform boundedness follows from Lemma 4.7.

To prove that  $(A_\epsilon + V_\epsilon)^{-1} \xrightarrow{CC} (A_0 + V_0)^{-1}$  we note that, for each sequence  $u_\epsilon \in U_\epsilon$  with  $\|u_\epsilon\|_{U_\epsilon} \leq 1$  we have

$$v_\epsilon = (A_\epsilon + V_\epsilon)^{-1}u_\epsilon = (I + A_\epsilon^{-1}V_\epsilon)^{-1}A_\epsilon^{-1}u_\epsilon$$

is a bounded sequence and that

$$v_\epsilon = -A_\epsilon^{-1}V_\epsilon v_\epsilon + A_\epsilon^{-1}u_\epsilon.$$

Taking subsequences we may assume that  $\{A_\epsilon^{-1}V_\epsilon v_\epsilon\}$  and  $\{A_\epsilon^{-1}u_\epsilon\}$  are convergent and it follows that  $\{v_\epsilon\}$  has a convergent subsequence. In addition, if  $\{u_\epsilon\}$  is convergent to  $u$  we have that from the above that  $\{v_\epsilon\}$  converges along subsequences to  $v$  which must satisfy

$$v = -A_0^{-1}V_0 v + A_0^{-1}u.$$

and  $v = (A_0 + V_0)^{-1}u$ . From the fact that the limit is independent of the subsequence we have convergence.  $\blacksquare$

Observing that, from Lemma 4.13,  $\bar{A}_\epsilon^{-1}$  converges compactly to  $\bar{A}_0^{-1}$  and proceeding exactly as in Lemma 4.11 we obtain the following result

**Corollary 4.14.** *Under the conditions of Proposition 4.13, all the results of Theorem 4.10 and Proposition 4.11, apply to the family of operators  $\bar{A}_\epsilon = A_\epsilon + V_\epsilon$ ,  $0 \leq \epsilon \leq 1$ .*

**Proof:** Just observe that from Proposition 4.13 the operators  $\bar{A}_\epsilon$  satisfy condition (4.5).  $\blacksquare$

## 5. CONTINUITY OF THE SET OF EQUILIBRIA

Let us consider in the family of Banach spaces  $U_\epsilon$  the following family of nonlinear problems

$$A_\epsilon u_\epsilon + f_\epsilon(u_\epsilon) = 0. \tag{5.1}$$

where  $f_\epsilon : U_\epsilon \rightarrow U_\epsilon$  is a bounded and differentiable map for  $\epsilon \in [0, 1]$ . Let  $\mathcal{E}_\epsilon = \{u_\epsilon^* : A_\epsilon u_\epsilon^* + f_\epsilon(u_\epsilon^*) = 0\}$ ,  $\epsilon \in [0, 1]$ .

We assume that

$$\boxed{A_\epsilon \text{ satisfies (4.5) and } A_\epsilon^{-1}f_\epsilon(\cdot) \xrightarrow{CC} A_0^{-1}f(\cdot).} \tag{5.2}$$

**Example 5.1.** Let  $\Omega_\epsilon$  be the dumbbell domain defined in Section 2 and consider the setting from Example 4.1 and Example 4.6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function with bounded derivatives up to second order. Let us show that if we denote by  $f_\epsilon : U_\epsilon^p \rightarrow U_\epsilon^p$ ,  $p > N$ , the Nemitskii map of  $f$  in  $U_\epsilon^p$ , then (5.2) is satisfied.

Suppose that  $U_\epsilon^p \ni u_\epsilon \xrightarrow{E} u \in U_0^p$ . Then,

$$\|f_\epsilon^e(u_\epsilon) - E_\epsilon f_0^e(u)\|_{U_\epsilon^p} = \|f_\epsilon^e(u_\epsilon) - f_\epsilon^e(E_\epsilon u)\|_{U_\epsilon^p} \leq L\|u_\epsilon - E_\epsilon u\|_{U_\epsilon^p} \xrightarrow{\epsilon \rightarrow 0} 0. \quad (5.3)$$

The condition (5.2) now follows from the compact convergence  $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$ , from the fact that  $f_\epsilon^e$  is bounded uniformly for  $\epsilon \in [0, 1]$  and from (5.3).

Consider the following definition of the index. We refer to [33] and [40] for details.

**Definition 5.2.** Let  $U$  be a real Banach space,  $\mathcal{O} \subset U$  and denote by  $\mathcal{K}(\Omega)$  the set of compact maps from  $\mathcal{O}$  into  $U$ . We say that a triple  $(I - F, \mathcal{O}, u)$  is admissible if  $\mathcal{O} \subset U$  is open and bounded,  $F \in \mathcal{K}(\mathcal{O})$  and  $u \notin (I - F)(\partial\mathcal{O})$ . A function  $\gamma$  which assigns an integer number  $\gamma(I - F, \mathcal{O}, u)$  to each admissible triple  $(I - F, \mathcal{O}, u)$  with the properties

- (1)  $\gamma(I, \mathcal{O}, u) = 1$  for  $u \in \mathcal{O}$ ;
- (2)  $\gamma(I - F, \mathcal{O}, u) = \gamma(I - F, \mathcal{O}_1, u) + \gamma(I - F, \mathcal{O}_2, u)$  whenever  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are disjoint open subsets of  $\mathcal{O}$  such that  $u \notin (I - F)(\bar{\mathcal{O}} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2))$ ;
- (3)  $\gamma(I - H(t, \cdot), \mathcal{O}, u(t))$  is independent of  $t \in [0, 1]$  whenever  $H : [0, 1] \times \bar{\mathcal{O}} \rightarrow U$  is compact,  $u(\cdot) : [0, 1] \rightarrow U$  is continuous and  $u(t) \notin (I - H(t, \cdot))(\partial\mathcal{O})$  on  $[0, 1]$ ;

is called a Leray-Schauder Degree.

Let  $F \in \mathcal{K}(\mathcal{O})$ ,  $u \in \mathcal{O}$  and  $\epsilon_0 > 0$ . If, for all  $\epsilon \in (0, \epsilon_0]$ ,  $(I - F, B_\epsilon(u), u)$  is an admissible triple and  $\gamma(I - F, B_\epsilon(u), u)$  is independent of  $\epsilon \in (0, \epsilon_0]$ , we say that this common value is the index of  $u$  relatively to the map  $I - F$  and denote it by  $\text{ind}(u, I - F)$ .

Now we can show,

**Theorem 5.3.** If  $u_0^*$  is an equilibrium point of (5.1) with  $\epsilon = 0$  which satisfies  $0 \notin \sigma(A_0 + f_0'(u_0^*))$  then,  $u_0^*$  is an isolated equilibrium point with  $|\text{ind}(u_0^*, I + A_0^{-1}f_0'(u_0^*))| = 1$ .

**Proof:** Note that  $u_0^*$  is a solution of (5.1) with  $\epsilon = 0$  if and only if it is a fixed point of the compact operator  $-A_0^{-1}f_0(\cdot) : U_0 \rightarrow U_0$ . Also,  $0 \notin \sigma(A_0 + f_0'(u_0^*))$  if and only if  $1 \notin \sigma(-A_0^{-1}f_0'(u_0^*))$ . It follows that there is a constant  $\eta > 0$  such that  $\|v + A_0^{-1}f_0'(u_0^*)v\|_{U_0} \geq 2\eta\|v\|_{U_0}$ . If we define  $w_0(u_0^*, v) = A_0^{-1}f_0(u_0^* + v) - A_0^{-1}f_0(u_0^*) - A_0^{-1}f_0'(u_0^*)v$ , then, by the differentiability of  $f_0$  we have that,

$$\frac{\|w_0(u_0^*, v)\|_{U_0}}{\|v\|_{U_0}} \xrightarrow{v \rightarrow 0} 0.$$

In particular, there is a  $r > 0$  such that  $\|w_0(u_0^*, v)\|_{U_0} \leq \eta\|v\|_{U_0}$  for  $\|v\|_{U_0} \leq r$ . Then, for  $\|u_0^* - u\|_{U_0} \leq r$  we have

$$\|u + A_0^{-1}f_0(u)\|_{U_0} = \|u - u_0^* - (A_0^{-1}f_0(u) - A_0^{-1}f_0(u_0^*))\|_{U_0} \geq$$

$$\|u - u_0^* + A_0^{-1}f_0'(u_0^*)(u - u_0^*)\|_{U_0} - \|w_0(u_0^*, u - u_0^*)\|_{U_0} \geq \eta\|u - u_0^*\|.$$

Thus  $u_0^*$  is an isolated equilibrium. The proof that  $|\text{ind}(u_0^*, I + A_0^{-1}f_0'(u_0^*))| = 1$  follows as a direct consequence of Theorem 21.6 in [33]. ■

**Corollary 5.4.** *If  $u_0^*$  is a hyperbolic solution of (5.1) with  $\epsilon = 0$  then,  $u_0^*$  is an isolated equilibrium and  $|\text{ind}(u_0^*, I + A_0^{-1}f_0'(u_0^*))| = 1$ .*

**Proposition 5.5.** *If all points in  $\mathcal{E}_0$  are isolated then, there is only a finite number of them. If  $0 \notin \sigma(A_0 + f_0'(u_0^*))$  for each  $u_0^* \in \mathcal{E}_0$  then,  $\mathcal{E}_0$  is a finite set with an odd number of elements.*

**Proof.** First we observe that all solutions of (5.1) with  $\epsilon = 0$  satisfies

$$u + A_0^{-1}f_0(u) = 0. \quad (5.4)$$

If we consider the ball of radius larger than  $\|A_0^{-1}\|K$ , with  $K = \sup\{\|f_0(u)\|_{U_0} : u \in U_0\}$ , then the operator  $-A_0^{-1}f(\cdot)$  maps the ball  $B(0, \|A_0^{-1}\|K) \subset U_0$  into itself. By Schauder fixed point Theorem (Theorem 21.6 in [33])  $\gamma(I + A_0^{-1}f(\cdot), \partial B(0, \|A_0^{-1}\|K)) = 1$  ( $\gamma$  is the Degree of Leray-Schauder) and there is at least one fixed point  $u_0^*$  for  $-A_0^{-1}f(\cdot)$  in  $B(0, \|A_0^{-1}\|K)$ ; that is,

$$u_0^* + A_0^{-1}f(u_0^*) = 0 \text{ with } u_0^* \in B(0, \|A_0^{-1}\|K).$$

Since the operator  $-A_0^{-1}f_0(\cdot) : U_0 \rightarrow U_0$  is compact we have that the set  $\mathcal{E}_0 = \{u : A_0u + f_0(u) = 0\}$  is compact in  $U_0$ . Moreover, by Theorem 5.3 any fixed point  $u_0^*$  is isolated. If the number of the fixed points is infinite, i.e. we have a sequence  $\{u_i^*\}_{i=1}^\infty$ , then the sequence  $-A_0^{-1}f(u_i^*) = u_i^* \rightarrow u_\infty^*$  converges on some subsequence  $i \in \mathbb{N}' \subset \mathbb{N}$ , which is a contradiction with the fact that each fixed point  $u_\infty^*$  is isolated. So the number of the equilibrium points is finite. Now by Theorem 20.6 in [33]

$$1 = \gamma(I + A_0^{-1}f(\cdot), \partial B(0, \|A_0^{-1}\|K)) = \sum_{i=1}^d \text{ind}(u_i^*, I + A_0^{-1}f(\cdot))$$

and therefore the number  $d = 2k + 1$  for some integer  $k \geq 0$ . ■

**Proposition 5.6.** *Assume that condition (5.2) is satisfied and that the problems (5.1) have solutions  $\{u_\epsilon^*\}$ ,  $\epsilon \in [0, 1]$ . Then, taking subsequences if necessary, there is a solution  $u_0^*$  of (5.1) with  $\epsilon = 0$  such that  $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .*

**Proof.** If  $u_\epsilon^*$  is a solution of (5.1) we have that  $u_\epsilon^* = -A_\epsilon^{-1}f_\epsilon(u_\epsilon^*)$ . From the fact that  $A_\epsilon^{-1}f_\epsilon(\cdot) : U_\epsilon \rightarrow U_\epsilon$  is bounded uniformly for  $\epsilon \in [0, 1]$  it follows that  $\{u_\epsilon^*\}$  is bounded. From (5.2), we have that there is a subsequence, which we again denote by  $u_\epsilon^*$ , such that  $u_\epsilon^* \xrightarrow{E} u_0^*$ . Again from (5.2) we have that

$$u_0^* + A_0^{-1}f^*(u_0^*) = 0,$$

which is equivalent to say that  $u_0^*$  is a solution of (5.1 with  $\epsilon = 0$ ). ■

**Proposition 5.7.** *Assume that (5.2) holds and that  $u_0^*$  is hyperbolic solution of (5.1) with  $\epsilon = 0$ . Then there are  $\epsilon_0$  and  $\delta > 0$  such that for  $0 < \epsilon \leq \epsilon_0$  the equations (5.1) have at least one solution  $u_\epsilon^*$  in  $\{w_\epsilon : \|w_\epsilon - E_\epsilon u_0^*\|_{U_\epsilon} \leq \delta\}$ . Furthermore,  $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .*

**Proof:** As in Corollary 5.4 there is a ball  $B(u_0^*, \delta)$  such that there are no other fixed points in it except  $u_0^*$  and we get  $|\text{ind}(u_0^*, I + A_0^{-1}f_0(\cdot))| = 1$ . It is easy to see that the hypotheses of Theorem 3 in [40] are satisfied and therefore there is at least one fixed point  $u_\epsilon^*$  in any ball  $B(E_\epsilon u_0^*, \delta)$ ,  $\epsilon \leq \epsilon_0$ , for some  $\epsilon_0 > 0$ . This sequence  $\{u_\epsilon^*\}$  is  $E$ -convergent to  $u_0^*$ . ■

The last two results, Proposition 5.6 and Proposition 5.7 show the continuity of the set of equilibria in the following sense: if  $u_\epsilon^*$  is a sequence of equilibria of (5.1) then we can

get a subsequence such that  $u_\epsilon^* \xrightarrow{E} u_0^*$ , which is an equilibrium of the limit equations and viceversa, if  $u_0^*$  is an equilibrium solution of the limit equation which is hyperbolic, then there exists a sequence of solutions  $u_\epsilon^*$  for all  $\epsilon > 0$  small enough such that  $u_\epsilon^* \xrightarrow{E} u_0^*$ .

We want to impose conditions now on the nonlinearities  $f_\epsilon$  that guarantee that for a fixed hyperbolic equilibrium solution  $u_0^*$  of the limit equation we have one and only one solution  $u_\epsilon^*$  of the perturbed equation nearby. In order to accomplish this, we will need some kind of uniform differentiability property of the nonlinearities  $f_\epsilon$ . For this, define first

$$w_\epsilon(u_\epsilon^*, v) = A_\epsilon^{-1} f_\epsilon(u_\epsilon^* + v) - A_\epsilon^{-1} f_\epsilon(u_\epsilon^*) - A_\epsilon^{-1} f'_\epsilon(u_\epsilon^*) v$$

Consider the following hypothesis

Hypothesis (5.2) holds, and if  $u_\epsilon^*$  are equilibrium solutions with  $u_\epsilon^* \xrightarrow{E} u_0^*$  then,  $A_\epsilon^{-1} f'_\epsilon(u_\epsilon^*) \xrightarrow{CC} A_0^{-1} f'_0(u_0^*)$  and  $\frac{\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon}}{\|v\|_{U_\epsilon}} = o(1)$  as  $\|v\|_{U_\epsilon} \rightarrow 0$ , uniformly in  $\epsilon$

(5.5)

Observe that saying that  $\frac{\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon}}{\|v\|_{U_\epsilon}} = o(1)$  uniformly in  $\epsilon$  means that for each  $\mu > 0$ , there exists a  $\delta > 0$  such that  $\|w_\epsilon(u_\epsilon^*, v)\|_{U_\epsilon} \leq \mu \|v\|_{U_\epsilon}$  for all  $v \in U_\epsilon$  with  $\|v\|_{U_\epsilon} \leq \delta$ .

We can show now the following

**Theorem 5.8.** *Assume (5.5) holds and let  $u_0^*$  be a solution of (5.1) with  $\epsilon = 0$  which satisfies  $0 \notin \sigma(A_0 + f'_0(u_0^*))$ . Then, there is a  $\delta > 0$  such that (5.1) has a unique solution  $u_\epsilon^*$  such that  $\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon} < \delta$ .*

*If, for all solutions  $u_0^*$  of (5.1) with  $\epsilon = 0$ ,  $0 \notin \sigma(A_0 + f'_0(u_0^*))$  then, from Proposition 5.5, (5.1) with  $\epsilon = 0$  has a finite number  $n_0$  of solutions  $u_1^*, \dots, u_{n_0}^*$ . In this case, there is an  $\epsilon_0$  such that (5.1) has exactly  $n_0$  solutions,  $u_{\epsilon,1}^*, \dots, u_{\epsilon,n_0}^*$ , for all  $\epsilon \leq \epsilon_0$  and  $u_{\epsilon,i}^* \xrightarrow{E} u_i^*$ ,  $1 \leq i \leq n_0$*

*If, moreover,  $u_0^*$  is a hyperbolic equilibrium point then  $u_\epsilon^*$  is also hyperbolic and we can apply Corollary 4.14. In particular, the linear unstable manifold of  $u_\epsilon^*$  E-converges to the linear unstable manifold of  $u_0^*$ .*

**Proof:** Note that  $u_\epsilon^*$  is a solution of (5.1) if and only if it is a fixed point of the compact operator  $-A_\epsilon^{-1} f_\epsilon(\cdot) : U_\epsilon \rightarrow U_\epsilon$ . Also, from Lemma 4.7, there is an  $\epsilon_0 > 0$  and  $\eta > 0$  (independent of  $\epsilon \in [0, \epsilon_0]$ ) such that, for any  $\epsilon \leq \epsilon_0$ ,  $0 \notin \sigma(A_\epsilon + f'_\epsilon(u_\epsilon^*))$  and  $\|(I + A_\epsilon^{-1} f'_\epsilon(u_\epsilon^*)) v_\epsilon\|_{U_\epsilon} \geq 2\eta \|v_\epsilon\|_{U_\epsilon}$ . If we write

$$A_\epsilon^{-1} f(u_\epsilon^* + v_\epsilon) - A_\epsilon^{-1} f(u_\epsilon^*) - A_\epsilon^{-1} f'_\epsilon(u_\epsilon^*) v_\epsilon = w_\epsilon(u_\epsilon^*, v_\epsilon), \quad \frac{\|w(u_\epsilon^*, v_\epsilon)\|_{U_\epsilon}}{\|v_\epsilon\|_{U_\epsilon}} \leq C \|v_\epsilon\|_{U_\epsilon}^{\frac{p-q}{q}}$$

with  $C$  independent of  $\epsilon$ . Hence, there is a  $\delta > 0$  (independent of  $\epsilon$ ) such that  $\|w_\epsilon(u_\epsilon^*, v_\epsilon)\|_{U_\epsilon} \leq \eta \|v_\epsilon\|_{U_\epsilon}$  for  $\|v_\epsilon\|_{U_0} \leq 2\delta$ . Then, for  $\|u_\epsilon^* - u_\epsilon\|_{U_\epsilon} \leq 2\delta$

$$\|u_\epsilon + A_\epsilon^{-1} f_\epsilon(u_\epsilon)\|_{U_\epsilon} \geq \|u_\epsilon - u_\epsilon^* + A_\epsilon^{-1} f'_\epsilon(u_\epsilon^*)(u_\epsilon^* - u_\epsilon)\|_{U_\epsilon} - \|w_\epsilon(u_\epsilon^*, u_\epsilon - u_\epsilon^*)\|_{U_\epsilon} \geq \eta \|u_\epsilon - u_\epsilon^*\|.$$

Thus  $u_\epsilon^*$  is the only solution of (5.1) in  $B_{2\delta}(u_\epsilon^*)$ . This together with the fact that  $u_\epsilon \xrightarrow{E} u_0^*$  implies the result. ■

**Example 5.9.** Assume we are exactly in the same conditions of Example 5.1. Let us show that hypotheses 5.5 also holds. Notice that if  $u_\epsilon^* \xrightarrow{E} u_0^*$ , and if we define  $V_\epsilon = f'(u_\epsilon^*)$ ,  $V_0 = f'(u_0)$ , we have that since  $f'$  is a bounded function that  $V_\epsilon \in L^\infty(\Omega_\epsilon)$ ,  $V_0 \in L^\infty(\Omega) \oplus L^\infty(0, 1)$ . Moreover,  $V_\epsilon \xrightarrow{E} V_0$ . Applying the results in Example 4.12, we get

$$A_\epsilon^{-1} f'(u_\epsilon^*) \xrightarrow{CC} A_0^{-1} f'(u_0^*)$$

Let us prove now that, for each  $\epsilon \in [0, 1]$ , we get

$$\|w_\epsilon(u_\epsilon, v)\|_{U_\epsilon^p} := \|A_\epsilon^{-1}(f_\epsilon^e(u_\epsilon + v) - f_\epsilon^e(u_\epsilon) - (f_\epsilon^e)'(u_\epsilon)v)\|_{U_\epsilon^p} \leq C \|v\|_{U_\epsilon^p}^{\frac{p}{q}}, \quad v \in U_\epsilon^p \quad (5.6)$$

for any  $N < q < p$ , where  $C$  is a constant independent of  $\epsilon$ . To prove (5.6) we note first that, as it will be proved in Section A, Lemma A.11, we have that for each  $N < q$  there exists a constant  $C$ , independent of  $\epsilon$ , such that

$$\|A_\epsilon^{-1}\|_{\mathcal{L}(U_\epsilon^q, L^\infty(\Omega_\epsilon))} \leq C \quad (5.7)$$

By interpolation, it is not difficult to see that if  $N < q < p$  we also have  $\|A_\epsilon^{-1}\|_{\mathcal{L}(U_\epsilon^q, U_\epsilon^p)} \leq C$ . Hence, if  $N < q < p$ , we have

$$\begin{aligned} \|A_\epsilon^{-1}(f_\epsilon^e(u_\epsilon + v) - f_\epsilon^e(u_\epsilon) - (f_\epsilon^e)'(u_\epsilon)v)\|_{U_\epsilon^p} &\leq C \|f(u_\epsilon + v) - f(u_\epsilon) - f'(u_\epsilon)v\|_{U_\epsilon^q} \\ &\leq C \|[f'(u_\epsilon(x) + \theta(x)v(x)) - f'(u_\epsilon(x))]v(x)\|_{U_\epsilon^q} \\ &\leq C \|f'(u_\epsilon(x) + \theta(x)v(x)) - f'(u_\epsilon(x))\|_{U_\epsilon^r} \|v(x)\|_{U_\epsilon^p}. \end{aligned} \quad (5.8)$$

where  $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$ . Note that

$$\begin{aligned} \|f'(u_\epsilon(x) + \theta(x)v(x)) - f'(u_\epsilon(x))\|_{L^\infty(\Omega)} &\leq C \\ \|f'(u_\epsilon(x) + \theta(x)v(x)) - f'(u_\epsilon(x))\|_{U_\epsilon^p} &\leq C \|v\|_{U_\epsilon^p} \end{aligned}$$

and by interpolation

$$\|f'(u_\epsilon(x) + \theta(x)v(x)) - f'(u_\epsilon(x))\|_{U_\epsilon^r} \leq C \|v\|_{U_\epsilon^p}^{\frac{p}{r}} \leq C \|v\|_{U_\epsilon^p}^{\frac{p-q}{q}}$$

which implies (5.6).

## 6. PROOF OF THE MAIN RESULTS: THEOREM 2.3 AND THEOREM 2.5

In this section we will assume that Proposition 2.7 is proved and will provide a demonstration of Theorem 2.3 and Theorem 2.5. The proof of Proposition 2.7 will be obtained in Appendix A.

**Proof of Theorem 2.3:** Under the conditions of the nonlinearity from Section 2 and with the aid of the maximum principle, we easily get that the set of equilibrium points  $\mathcal{E}_\epsilon$  is bounded in  $L^\infty(\Omega_\epsilon)$  with a bound independent of  $\epsilon$ . Similarly, the set of equilibria of the limit problem is also uniformly bounded.

Notice that, if  $p > N$ , with the definitions of  $U_\epsilon^p$  and  $U_0^p$  from Section 2 and Example 4.1, we have from Proposition 2.7 the compact convergence of  $A_\epsilon^{-1}$  to  $A_0^{-1}$ . In particular, (4.5) holds true. Moreover, as it is shown in Example 5.1, condition (5.2) is also satisfied. Applying now Proposition 5.6, we show (2.7).



If we denote now by  $f_\epsilon^* = f(u_\epsilon^*) \in U_\epsilon^p$  and  $f_0^* = f(u_0^*) \in U_0^p$ , by (2.7) and by the continuity of the nonlinearity  $f$ , we have that  $\|f_\epsilon^* - E_\epsilon f_0^*\|_{U_\epsilon^p} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Applying Proposition 2.7, point (3) i), ii) and iii) and taking into account that  $u_\epsilon^* = A_\epsilon^{-1} f_\epsilon^*$ ,  $u_0^* = A_0^{-1} f_0^*$ , we prove i) of Theorem 2.3.

To show ii), observe that by Example 5.9, we have that hypothesis (5.5) holds true. In particular, we can apply Theorem 5.8, which proves ii). This concludes the proof of the theorem.  $\blacksquare$

**Proof of Theorem 2.5:** If we are in the conditions of Theorem 2.3 and we have a sequence of equilibria  $u_\epsilon^*$  which  $E$ -converges to  $u_0^* = (w_0, v_0)$  satisfying (2.7) and (2.8), we have that if we define  $V_\epsilon = f'(u_\epsilon^*) + M$  and  $V_0 = f'(u_0^*) + M$ , for some positive  $M$  large enough so that  $f'(u_0^*) + M \geq 0$ , then, as it is shown in Example 4.12, (4.13) holds. Moreover, since  $f'(u_0^*) + M \geq 0$ , we have that (4.14) also holds.

Hence, we can apply Proposition 4.13 which in particular implies that the spectral convergence result given by Theorem 4.10 hold true for the operators  $A_\epsilon + f'(u_\epsilon^*) + M$  and  $A_0 + f'(u_0^*) + M$ . Since the effect of the constant  $M$  in the operators above is just a shift in the spectrum, we show that the results of Theorem 4.10 hold true for the operators  $A_\epsilon + f'(u_\epsilon^*)$  and  $A_0 + f'(u_0^*)$ . In particular, we obtain the convergence of the eigenvalues and the convergence of the spectral projections in  $U_\epsilon^p$ . To show the convergence in the  $H_\epsilon^1$  norm we proceed similarly as in Theorem 2.3.  $\blacksquare$

## APPENDIX A. RESOLVENT CONVERGENCE

In this Appendix we will show Proposition 2.7, which is the main result on the convergence of the resolvent operators.

Before we start comparing the resolvent operators of  $A_\epsilon$  and  $A_0$ , we present some preliminary results, including some extension and projection operators, that will be needed to prove the result.

**A.1. The Projection.** In this subsection we present the basic projection operator that we will use.

Let  $\psi_\epsilon \in U_\epsilon^p$  where  $U_\epsilon^p = L^p(\Omega_\epsilon)$  with the norm

$$\|\phi_\epsilon\|_{U_\epsilon} = \|\phi_\epsilon\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|\phi_\epsilon\|_{L^p(R_\epsilon)},$$

for  $\epsilon > 0$  and  $U_0^p = L^p(\Omega) \oplus L_g^p(0, 1)$  with the norm

$$\|(w, v)\|_{U_0^p} = \|w\|_{L^p(\Omega)} + \|v\|_{L_g^p(0, 1)}.$$

where  $\|w\|_{L_g^p(0, 1)} = (\int_0^1 |w(s)|^p g(s) ds)^{1/p}$ .

To compare functions from  $U_\epsilon^p$  and from  $U_0^p$ , we define the following projection operator

$$\begin{aligned} M_\epsilon : U_\epsilon^p &\rightarrow U_0^p \\ \psi_\epsilon &\rightarrow (M_\epsilon \psi_\epsilon)(x) = \begin{cases} \psi_\epsilon(x), & x \in \Omega \\ T_\epsilon^s \psi_\epsilon, & s \in (0, 1), \end{cases} \end{aligned} \quad (\text{A.1})$$

where

$$\begin{aligned} T_\epsilon^s \psi_\epsilon(x) &= \frac{1}{|\Gamma_\epsilon^s|} \int_{\Gamma_\epsilon^s} \psi(s, y) dy \\ \Gamma_\epsilon^s &= \{y : (s, y) \in R_\epsilon\}. \end{aligned} \quad (\text{A.2})$$

The following result holds

**Lemma A.1.** *The projection  $M_\epsilon$  is a bounded operator with norm  $\|M_\epsilon\|_{L(U_\epsilon, U_0^p)} = 1$ .*

**Proof:** If  $\phi_\epsilon \in U_\epsilon$  then, if  $x = (s, y)$  with  $s \in \mathbb{R}$  and  $y \in \mathbb{R}^{N-1}$ ,

$$\begin{aligned} \|M_\epsilon \phi_\epsilon\|_{U_0^p} &= \left( \int_\Omega |\phi_\epsilon(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_0^1 g(s) |M_\epsilon \phi_\epsilon(s)|^p ds \right)^{\frac{1}{p}} \\ &= \left( \int_\Omega |\phi_\epsilon(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_0^1 g(s) \left| \frac{1}{|\Gamma_\epsilon^s|} \int_{\Gamma_\epsilon^s} \phi_\epsilon(s, y) dy \right|^p ds \right)^{\frac{1}{p}} \\ &= \left( \int_\Omega |\phi_\epsilon(x)|^p dx \right)^{\frac{1}{p}} + \epsilon^{1-N} \left( \int_0^1 g(s)^{-p+1} \left| \int_{\Gamma_\epsilon^s} \phi_\epsilon(s, y) dy \right|^p ds \right)^{\frac{1}{p}} \\ &\leq \left( \int_\Omega |\phi_\epsilon(x)|^p dx \right)^{\frac{1}{p}} + \epsilon^{1-N} \left( \int_0^1 g(s)^{-p+1} |\Gamma_\epsilon^s|^{p-1} \int_{\Gamma_\epsilon^s} |\phi_\epsilon(s, y)|^p dy ds \right)^{\frac{1}{p}} \\ &= \left( \int_\Omega |\phi_\epsilon(x)|^p dx \right)^{\frac{1}{p}} + \epsilon^{\frac{1-N}{p}} \left( \int_0^1 \int_{\Gamma_\epsilon^s} |\phi_\epsilon(s, y)|^p dy ds \right)^{\frac{1}{p}} = \|\phi_\epsilon\|_{U_\epsilon}. \end{aligned}$$

The equality holds if  $\phi_\epsilon$  is independent of  $y$  in  $R_\epsilon$ . ■

**A.2. The Extension.** Let  $\psi \in U_0^p$ , to consider  $\psi$  as a function in  $U_\epsilon^p$ , we define the following extension operator

$$\begin{aligned} E_\epsilon : U_0^p &\rightarrow U_\epsilon \\ \psi &\rightarrow (E_\epsilon \psi)(x) = \begin{cases} \psi(x), & \Omega \\ \psi(s), & (s, y) \in R_\epsilon. \end{cases} \end{aligned} \quad (\text{A.3})$$

Of course  $E_\epsilon$  can be considered in larger spaces with the same definition. It is easy to see that  $E_\epsilon$  has the following property

**Lemma A.2.**  $E_\epsilon : U_0^p \rightarrow U_\epsilon^p$  is a bounded linear operator and

$$\|E_\epsilon(w, v)\|_{U_\epsilon^p} = \|(w, v)\|_{U_0^p},$$

for all  $(w, v) \in U_0^p$ .

**Lemma A.3.** *There is a positive constant  $C$  such that, for  $\psi_\epsilon \in H^1(\Omega_\epsilon)$  we have that*

$$\|\psi_\epsilon\|_{L^2(R_\epsilon)}^2 = \|\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon\|_{L^2(R_\epsilon)}^2 + \|E_\epsilon M_\epsilon \psi_\epsilon\|_{L^2(R_\epsilon)}^2 \quad (\text{A.4})$$

$$\|E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon\|_{L^2(R_\epsilon)}^2 \leq C \epsilon^2 \left\| \frac{\partial \psi_\epsilon}{\partial y} \right\|_{L^2(R_\epsilon)}^2. \quad (\text{A.5})$$

**Proof:** Note that

$$\begin{aligned} \|\psi_\epsilon\|_{L^2(\mathbb{R}_\epsilon)}^2 &= \int_{\mathbb{R}_\epsilon} |\psi_\epsilon|^2 dx = \int_{\mathbb{R}_\epsilon} |(\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) + E_\epsilon M_\epsilon \psi_\epsilon|^2 dx \\ &= \int_{\mathbb{R}_\epsilon} |\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon|^2 + 2 \int_{\mathbb{R}_\epsilon} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon + \int_{\mathbb{R}_\epsilon} |E_\epsilon M_\epsilon \psi_\epsilon|^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{\mathbb{R}_\epsilon} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon &= \int_0^1 \int_{\Gamma_\epsilon^s} (\psi_\epsilon - E_\epsilon M_\epsilon \psi_\epsilon) E_\epsilon M_\epsilon \psi_\epsilon ds dy \\ &= \int_0^1 M_\epsilon \psi_\epsilon(x) \left\{ \int_{\Gamma_\epsilon^s} [\psi_\epsilon(x) - E_\epsilon M_\epsilon \psi_\epsilon(x)] dy \right\} ds = 0. \end{aligned}$$

And the identity (A.4) follows. Observe that

$$\|E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon\|_{L^2(\mathbb{R}_\epsilon)}^2 = \int_{\mathbb{R}_\epsilon} |(E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)|^2 dx = \int_0^1 \int_{\Gamma_\epsilon^s} |(E_\epsilon M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)|^2 ds dy.$$

Hence, let us estimate  $\int_{\Gamma_\epsilon^s} |(M_\epsilon \psi_\epsilon - \psi_\epsilon)(x)|^2 dy$ . In fact, from the variational characterization of eigenvalues for the Neumann Laplacian in  $\Gamma_\epsilon^s$ , we have that

$$\lambda_2(\Gamma_\epsilon^s) = \min \left\{ \frac{\int_{\Gamma_\epsilon^s} |\nabla \phi|^2}{\int_{\Gamma_\epsilon^s} |\phi|^2} : \phi \in H^1(\Gamma_\epsilon^s), \phi \neq 0, \int_{\Gamma_\epsilon^s} \phi = 0 \right\}. \quad (\text{A.6})$$

Taking  $\phi = M_\epsilon \psi_\epsilon - \psi_\epsilon$ , we have

$$\int_{\Gamma_\epsilon^s} |M_\epsilon \psi_\epsilon - \psi_\epsilon|^2 \leq \frac{1}{\lambda_2(\Gamma_\epsilon^s)} \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2. \quad (\text{A.7})$$

From (A.6), it follows that

$$\begin{aligned} \lambda_2(\Gamma_\epsilon^s) &= \min \left\{ \frac{\int_{\Gamma_\epsilon^s} |\nabla \phi|^2}{\int_{\Gamma_\epsilon^s} |\phi|^2} : \phi \in H^1(\Gamma_\epsilon^s), \phi \neq 0, \int_{\Gamma_\epsilon^s} \phi = 0 \right\} \\ &= \frac{1}{\epsilon^2} \min \left\{ \frac{\int_{\Gamma_1^s} |\nabla \tilde{\phi}|^2}{\int_{\Gamma_1^s} |\tilde{\phi}|^2} : \tilde{\phi} \in H^1(\Gamma_1^s), \tilde{\phi} \neq 0, \int_{\Gamma_1^s} \tilde{\phi} = 0 \right\} = \frac{1}{\epsilon^2} \lambda_2(\Gamma_1^s), \end{aligned} \quad (\text{A.8})$$

where  $\lambda_2(\Gamma_1^s)$  is the second eigenvalue of the Neumann Laplacian in  $\Gamma_1^s$ . Using (A.8) and (A.7), we have that

$$\int_{\Gamma_\epsilon^s} |M_\epsilon \psi_\epsilon - \psi_\epsilon|^2 \leq \frac{\epsilon^2}{\lambda_2(\Gamma_1^s)} \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2 \leq C \epsilon^2 \int_{\Gamma_\epsilon^s} \left| \frac{\partial \psi_\epsilon}{\partial y} \right|^2, \quad (\text{A.9})$$

where we used the fact that the map  $[0, 1] \ni s \rightarrow \lambda_2(\Gamma_1^s) \in (0, \infty)$  is continuous and therefore attains its minimum at a positive value; that is,

$$m := \min_{0 \leq s \leq 1} \lambda_2(\Gamma_1^s) = \lambda_2(\Gamma_1^{\bar{x}}) > 0, \quad \bar{x} \in [0, 1],$$

from which we have  $C := \frac{1}{m} \geq \frac{1}{\lambda_2(\Gamma_1^s)}$ . Now, integrating from 0 to 1 we have the inequality (A.5). ■

**A.3. Continuous Extension.** Observe that the operator  $E_\epsilon$  does not takes continuous functions into continuous functions. When such property is required we consider the following extension operator. If  $\mathcal{C} = \{(w, v) \in C(\bar{\Omega}) \oplus C(0, 1) \text{ with } w(0) = v(0) \text{ and } w(1) = v(1)\}$  then

$$\begin{aligned} E_\epsilon^\epsilon : \quad \mathcal{C} &\rightarrow C(\bar{\Omega}_\epsilon) \\ (w, v) &\rightarrow E_\epsilon^\epsilon(w, v) = \begin{cases} w, & x \in \Omega \\ \tilde{v}, & x \in R_\epsilon, \end{cases} \end{aligned} \quad (\text{A.10})$$

where

$$\tilde{v}(x) = v(s) + h_\epsilon(s) (w(0, y) - v(0)) + h_\epsilon(1 - s) (w(1, y) - v(1)), \quad x \in R_\epsilon, \quad (\text{A.11})$$

$h_\delta(s) = h(\frac{s}{\delta})$ ,  $h : \mathbb{R}^+ \rightarrow [0, 1]$  is  $C^\infty$  function such that

$$h(s) = \begin{cases} 1, & \text{for } s \in [0, 1/4], \\ 0, & \text{for } s \geq 3/4 \end{cases}$$

and  $|h'(s)| \leq C$ .

We can easily estimate the difference of these operators in the following way.

**Lemma A.4.** *Let  $E_\epsilon$  and  $E_\epsilon^\epsilon$  be the extension operators defined above. If  $(w, v) \in \mathcal{C}$  with  $(w, v) \in C^1(\bar{\Omega}) \oplus C^1([0, 1])$  we have that*

$$\|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(\Omega_\epsilon)} \leq C\epsilon^{\frac{N+2}{2}} \|w\|_{C^1(\bar{\Omega})} \quad (\text{A.12})$$

$$\|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{H^1(\Omega) \oplus H^1(R_\epsilon)} \leq C\epsilon^{N/2} \|w\|_{C^1(\bar{\Omega})} \quad (\text{A.13})$$

$$\left| \|E_\epsilon^\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 - \|E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 \right| \leq C\epsilon^{N+1} \|w\|_{C^1(\bar{\Omega})} \|v\|_{C^0(0,1)} \quad (\text{A.14})$$

$$\left| \|\nabla E_\epsilon^\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 - \|\nabla E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 \right| \leq C\epsilon^N \|w\|_{C^1(\bar{\Omega})} \|v\|_{C^1(0,1)}. \quad (\text{A.15})$$

**Proof:** Let  $\mathcal{C} \ni (w, v) \in C^1(\bar{\Omega}) \oplus C(0, 1)$  then, since  $\|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(\Omega_\epsilon)}^2 = \|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(\Omega)}^2 + \|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(R_\epsilon)}^2 = \|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(R_\epsilon)}^2$ , we only need to estimate  $\|(E_\epsilon - E_\epsilon^\epsilon)(w, v)\|_{L^2(R_\epsilon)}^2$ .

$$\begin{aligned}
\|(E_\epsilon - E_C^\epsilon)(w, v)\|_{L^2(R_\epsilon)}^2 &= \int_0^\epsilon \int_{\Gamma_\epsilon^s} |h_\epsilon(s)(w(0, y) - v(0))|^2 dy ds \\
&+ \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} |h_\epsilon(1-s)(w(1, y) - v(1))|^2 dy ds \\
&= \int_0^\epsilon \int_{\Gamma_\epsilon^s} |h_\epsilon(s)|^2 |w(0, y) - w(0, 0)|^2 dy ds \\
&+ \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} |h_\epsilon(1-s)|^2 |w(1, y) - w(1, 0)|^2 dy ds \\
&\leq C_1 \int_0^\epsilon \int_{\Gamma_\epsilon^s} |y|^2 \left( \sup_{y \in \Gamma_\epsilon^s} \left| \frac{\partial w}{\partial y} \right| \right)^2 dy ds + C_1 \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} |y|^2 \left( \sup_{y \in \Gamma_\epsilon^s} \left| \frac{\partial w}{\partial y} \right| \right)^2 dy ds \\
&\leq 2C_1 \left( \int_0^\epsilon \int_{\Gamma_\epsilon^s} |y|^2 dy ds + \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} |y|^2 dy ds \right) \|w\|_{C^1(\bar{\Omega})}^2 \leq C\epsilon^{N+2} \|w\|_{C^1(\bar{\Omega})}^2.
\end{aligned} \tag{A.16}$$

where we have used the fact that  $v(0) = w(0, 0)$  and  $v(1) = w(1, 0)$  and that  $|\Gamma_\epsilon^s| \leq C\epsilon^{N-1}$ . This shows (A.12).

To show (A.13), it is enough to estimate  $\|\nabla(E_\epsilon - E_C^\epsilon)(w, v)\|_{L^2(R_\epsilon)}$ . Since  $h'_\epsilon(s) = \epsilon^{-1}h'(s/\epsilon)$  and with a similar argument we have

$$\begin{aligned}
\|\nabla(E_\epsilon - E_C^\epsilon)(w, v)\|_{L^2(R_\epsilon)}^2 &= \int_0^\epsilon |h'_\epsilon(s)|^2 \int_{\Gamma_\epsilon^s} |w(0, y) - v(0)|^2 dy ds \\
&+ \int_0^\epsilon |h_\epsilon(s)|^2 \int_{\Gamma_\epsilon^s} |\nabla_y w(0, y)|^2 dy ds \\
&+ \int_{1-\epsilon}^1 |h'_\epsilon(1-s)|^2 \int_{\Gamma_\epsilon^s} |w(1, y) - v(1)|^2 dy ds \\
&+ \int_{1-\epsilon}^1 |h_\epsilon(1-s)|^2 \int_{\Gamma_\epsilon^s} |\nabla_y w(1, y)|^2 dy ds \\
&\leq C_1 \epsilon^{-2} \int_0^\epsilon \int_{\Gamma_\epsilon^s} |y|^2 dy ds \|w\|_{C^1(\bar{\Omega})}^2 + \tilde{C}_1 \int_0^\epsilon \int_{\Gamma_\epsilon^s} dy ds \|w\|_{C^1(\bar{\Omega})}^2 \\
&+ C_1 \epsilon^{-2} \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} |y|^2 dy ds \|w\|_{C^1(\bar{\Omega})}^2 + \tilde{C}_1 \int_{1-\epsilon}^1 \int_{\Gamma_\epsilon^s} dy ds \|w\|_{C^1(\bar{\Omega})}^2 \\
&\leq 2C_1 \epsilon^N \|w\|_{C^1(\bar{\Omega})}^2 + 2\tilde{C}_1 \epsilon^N \|w\|_{C^1(\bar{\Omega})}^2 \leq C\epsilon^N \|w\|_{C^1(\bar{\Omega})}^2,
\end{aligned}$$

where we have also used that  $\int_0^\epsilon \int_{\Gamma_\epsilon^s} ds dy = O(\epsilon^N)$ .

The proof of the last two inequalities follows from the previous in the following way

$$\begin{aligned}
\|E_C^\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 &= \|E_C^\epsilon(w, v) - E_\epsilon(w, v) + E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 = \|E_C^\epsilon(w, v) - E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 \\
&+ \|E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 + 2(E_C^\epsilon(w, v) - E_\epsilon(w, v), E_\epsilon(w, v))_{L^2(R_\epsilon)}
\end{aligned} \tag{A.17}$$

and

$$\begin{aligned} \|\nabla E_C^\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 &= \|\nabla E_C^\epsilon(w, v) - \nabla E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 + \|\nabla E_\epsilon(w, v)\|_{L^2(R_\epsilon)}^2 \\ &\quad + 2(\nabla E_C^\epsilon(w, v) - \nabla E_\epsilon(w, v), \nabla E_\epsilon(w, v))_{L^2(R_\epsilon)} \end{aligned} \quad (\text{A.18})$$

But, taking into account that  $E_C^\epsilon(w, v) = E_\epsilon(w, v)$  apart from the set  $\tilde{R}_\epsilon = \{(s, y) \in R_\epsilon : 0 < s < \epsilon, 1 - \epsilon < s < 1\}$  which has measure of the order of  $\epsilon^N$ , then

$$\begin{aligned} |(E_C^\epsilon(w, v) - E_\epsilon(w, v), E_\epsilon(w, v))_{L^2(R_\epsilon)}| &\leq \|E_C^\epsilon(w, v) - E_\epsilon(w, v)\|_{L^2(R_\epsilon)} \|E_\epsilon(w, v)\|_{L^2(\tilde{R}_\epsilon)} \\ &\leq C\epsilon^{\frac{N+2}{2}} \|w\|_{C^1(\bar{\Omega})} \epsilon^{\frac{N}{2}} \|v\|_{C(0,1)} \leq C\epsilon^{N+1} \|w\|_{C^1(\bar{\Omega})} \|v\|_{C(0,1)} \end{aligned}$$

and with a similar argument

$$\begin{aligned} |(\nabla E_C^\epsilon(w, v) - \nabla E_\epsilon(w, v), \nabla E_\epsilon(w, v))_{L^2(R_\epsilon)}| &\leq \left\| \frac{\partial E_C^\epsilon(w, v)}{\partial s} - \frac{\partial E_\epsilon(w, v)}{\partial s} \right\|_{L^2(R_\epsilon)} \left\| \frac{\partial E_\epsilon(w, v)}{\partial s} \right\|_{L^2(\tilde{R}_\epsilon)} \\ &\leq C\epsilon^N \|w\|_{C^1(\bar{\Omega})} \|v\|_{C^1(0,1)}, \end{aligned}$$

which proves the lemma. ■

**A.4. Some Auxiliary Lemmas.** Denote by  $p_0 = (0, \mathbf{0})$ ,  $p_1 = (1, \mathbf{0})$ ,  $\mathbf{0} \in \mathbb{R}^{N-1}$  and  $p$  a generic point in  $\mathbb{R}^N$ .  $B(p, \rho)$  is the ball of radius  $\rho$  around  $p$ . Consider the following:

$$\begin{aligned} B_\rho^l(p) &= B(p, l) \setminus \overline{B(p, \rho)}, \quad \text{for } \rho < l, \\ D_\rho^L &= B(p_0, \rho) \cap \Omega, \quad \text{for } 0 < \rho \leq l, \\ D_\rho^R &= B(p_1, \rho) \cap \Omega, \quad \text{for } 0 < \rho \leq l, \\ S_\rho^L &= \Omega \cap \overline{B_\rho^l(p_0)}, \quad S_\rho^R = \Omega \cap \overline{B_\rho^l(p_1)}, \\ \tilde{\Omega}^L &= \Omega \setminus \overline{B(p_0, l)}, \quad \tilde{\Omega}^R = \Omega \setminus \overline{B(p_1, l)}, \\ \hat{\Gamma}_\rho^0 &= \{(s, y) : |s|^2 + |y|^2 = \rho, s < 0\}, \\ \hat{\Gamma}_\rho^1 &= \{(s, y) : |s - 1|^2 + |y|^2 = \rho, s > 1\} \\ (\phi, \psi)_{L^2(\Omega)} &= \int_\Omega \phi \cdot \psi, \quad (\phi, \psi)_{L^2(R_\epsilon)} = \int_{L^2(R_\epsilon)} \phi \cdot \psi. \end{aligned} \quad (\text{A.19})$$

For a function  $\psi_\epsilon$  defined in  $\Omega_\epsilon$ , we write

$$\hat{T}_\rho^0 \psi_\epsilon = \frac{1}{|\hat{\Gamma}_\rho^0|} \int_{\hat{\Gamma}_\rho^0} \psi_\epsilon, \quad \hat{T}_\rho^1 \psi_\epsilon = \frac{1}{|\hat{\Gamma}_\rho^1|} \int_{\hat{\Gamma}_\rho^1} \psi_\epsilon.$$

With this we have the following result.

**Lemma A.5.** *Let  $r \geq 1$ ,  $T_\epsilon^0, T_\epsilon^1$  as in (A.2),  $\hat{T}_{\epsilon r}^0, \hat{T}_{\epsilon r}^1$  as above and  $\psi_\epsilon \in H^1(\Omega)$ . Then, there is a constant  $C = C(N)$  such that*

$$\begin{aligned} |T_\epsilon^0 \psi_\epsilon - \hat{T}_{\epsilon r}^0 \psi_\epsilon| &\leq C(N) \epsilon^{(-N+2)/2} \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^L} \\ |T_\epsilon^1 \psi_\epsilon - \hat{T}_{\epsilon r}^1 \psi_\epsilon| &\leq C(N) \epsilon^{(-N+2)/2} \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^R} \end{aligned} \quad (\text{A.20})$$

**Proof:** We only prove the inequality for  $|T_\epsilon^0 \psi_\epsilon - \widehat{T}_{\epsilon r}^0 \psi_\epsilon|$ . The other is similar. Observe that  $\epsilon D_r^L = D_{\epsilon r}^L$  e  $\epsilon \Gamma_1^0 = \Gamma_\epsilon^0$ . If  $\psi_\epsilon \in H^1(\Omega)$ , we define  $\bar{\psi}_\epsilon(x, y) = \psi_\epsilon(\epsilon x, \epsilon y)$  para  $(x, y) \in D_r^L$  and also  $a_\epsilon = \frac{1}{|D_{\epsilon r}^L|} \int_{D_{\epsilon r}^L} \psi_\epsilon = \frac{1}{|D_r^L|} \int_{D_r^L} \bar{\psi}_\epsilon$ . Thus,

$$\begin{aligned} |a_\epsilon - T_\epsilon^0 \psi_\epsilon| &= \frac{1}{|\Gamma_\epsilon^0|} \left| \int_{\Gamma_\epsilon^0} (a_\epsilon - \psi_\epsilon(0, y)) dy \right| = \left| \frac{1}{|\Gamma_1^0|} \int_{\Gamma_1^0} (a_\epsilon - \bar{\psi}_\epsilon(0, y)) dy \right| \\ &\leq \frac{1}{|\Gamma_1^0|} \int_{\Gamma_1^0} |a_\epsilon - \bar{\psi}_\epsilon| \leq C \|a_\epsilon - \bar{\psi}_\epsilon(0, \cdot)\|_{L^2(\Gamma_1^0)} \\ &\leq C \|a_\epsilon - \bar{\psi}_\epsilon\|_{H^1(D_r^L)} = C \|a_\epsilon - \bar{\psi}_\epsilon\|_{L^2(D_r^L)} + C \|\nabla \bar{\psi}_\epsilon\|_{L^2(D_r^L)}. \end{aligned}$$

Now, the Poincaré inequality  $\|a_\epsilon - \bar{\psi}_\epsilon\|_{L^2(D_r^L)} \leq C \|\nabla \bar{\psi}_\epsilon\|_{L^2(D_r^L)}$ , implies that  $|a_\epsilon - T_\epsilon^0 \psi_\epsilon| \leq C \|\nabla \bar{\psi}_\epsilon\|_{L^2(D_r^L)}$ . Since  $\|\nabla \bar{\psi}_\epsilon\|_{L^2(D_r^L)}^2 = \epsilon^{-N+2} \|\nabla \psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}^2$  we conclude that  $|a_\epsilon - T_\epsilon^0 \psi_\epsilon| \leq \epsilon^{(-N+2)/2} \|\nabla \psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}$ .

Following the same reasoning as above we obtain  $|a_\epsilon - \widehat{T}_\epsilon^0 \psi_\epsilon| \leq \epsilon^{(-N+2)/2} \|\nabla \psi_\epsilon\|_{L^2(D_{\epsilon r}^L)}$  and therefore

$$|T_\epsilon^0 \psi_\epsilon - \widehat{T}_\epsilon^0 \psi_\epsilon| \leq |T_\epsilon^0 \psi_\epsilon - a_\epsilon| + |a_\epsilon - \widehat{T}_\epsilon^0 \psi_\epsilon|.$$

Hence, using  $|T_\epsilon^0 \psi_\epsilon - a_\epsilon|$  and  $|a_\epsilon - \widehat{T}_\epsilon^0 \psi_\epsilon|$  in the previous inequality, we conclude the proof of the lemma.  $\blacksquare$

**Lemma A.6.** *There is a constant  $C = C(N)$  such that, if  $\psi_\epsilon \in H^1(\Omega)$  then,*

$$\left( T_\epsilon^0 \psi_\epsilon - \widehat{T}_r^0 \psi_\epsilon \right)^2 + \left( T_\epsilon^1 \psi_\epsilon - \widehat{T}_r^1 \psi_\epsilon \right)^2 \leq \begin{cases} C(2) |\ln \epsilon| \|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N = 2 \\ C(N) \epsilon^{2-N} \|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N > 2 \end{cases} \quad (\text{A.21})$$

**Proof:** We prove the lemma for  $N > 2$ . For the case  $N = 2$ , we refer to [4]. For  $i = 0, 1$ , consider the operators  $\widehat{T}_\rho^i \psi_\epsilon$  as above,  $0 < \rho \leq r$ .

We have that,

$$\|\nabla \psi_\epsilon\|_\Omega^2 \geq \|\nabla \psi_\epsilon\|_{D_r^L}^2 + \|\nabla \psi_\epsilon\|_{D_r^R}^2 = \|\nabla \psi_\epsilon\|_{S_{\epsilon r}^L}^2 + \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^L}^2 + \|\nabla \psi_\epsilon\|_{S_{\epsilon r}^R}^2 + \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^R}^2.$$

But,

$$\begin{aligned} \|\nabla \psi_\epsilon\|_{S_{\epsilon r}^L}^2 &\geq \min \left\{ \|\nabla \chi_\epsilon\|_{S_{\epsilon r}^L}^2 : \widehat{T}_l^0 \chi_\epsilon = \widehat{T}_r^0 \psi_\epsilon, \widehat{T}_{\epsilon r}^0 \chi_\epsilon = \widehat{T}_{\epsilon r}^0 \psi_\epsilon \right\} \\ &= \left( \widehat{T}_{\epsilon r}^0 \psi_\epsilon - \widehat{T}_r^0 \psi_\epsilon \right)^2 \min \left\{ \|\nabla \tilde{\chi}_\epsilon\|_{S_{\epsilon r}^L}^2 : \widehat{T}_r^0 \tilde{\chi}_\epsilon = 0, \widehat{T}_{\epsilon r}^0 \tilde{\chi}_\epsilon = 1 \right\} \\ &= \left( \widehat{T}_{\epsilon r}^0 \psi_\epsilon - \widehat{T}_r^0 \psi_\epsilon \right)^2 C(N) \epsilon^{N-2} (1 + o(1)) \end{aligned}$$

where we have used that the minimum is attained when  $\tilde{\chi}_\epsilon$  is the solution of

$$\begin{cases} -\Delta \tilde{\chi}_\epsilon = 0, & S_{\epsilon r}^L \\ \widehat{T}_r^0 \tilde{\chi}_\epsilon = 0, & \widehat{T}_{\epsilon r}^0 \tilde{\chi}_\epsilon = 1, \end{cases}$$

$$\text{and } \min \left\{ \|\nabla \chi\|_{S_\eta^L}^2 : \widehat{T}_r^0 \chi = 0, \widehat{T}_\eta^0 \chi = 1 \right\} = C(N) \frac{\eta^{N-2}}{1 - \eta^{N-2}}.$$

From Lemma A.5, we have that

$$\|\nabla \psi_\epsilon\|_{D_{\epsilon r}^L}^2 \geq C(N) \epsilon^{N-2} \left( T_\epsilon^0 \psi_\epsilon - \widehat{T}_{\epsilon r}^0 \psi_\epsilon \right)^2.$$

Therefore

$$\begin{aligned} \|\nabla \psi_\epsilon\|_{S_{\epsilon r}^L}^2 + \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^L}^2 &\geq C(N) \left[ \left( \widehat{T}_r^0 \psi_\epsilon - \widehat{T}_{\epsilon r}^0 \psi_\epsilon \right)^2 + \left( \widehat{T}_{\epsilon r}^0 \psi_\epsilon - T_\epsilon^0 \psi_\epsilon \right)^2 \right] \epsilon^{N-2} (1 + o(1)) \\ &\geq \frac{1}{2} C(N) \left( \widehat{T}_r^0 \psi_\epsilon - T_\epsilon^0 \psi_\epsilon \right)^2 \epsilon^{N-2} (1 + o(1)) \end{aligned}$$

With a similar reasoning we obtain an estimate for  $\|\nabla \psi_\epsilon\|_{S_{\epsilon r}^R}^2 + \|\nabla \psi_\epsilon\|_{D_{\epsilon r}^R}^2$ . This concludes the proof.  $\blacksquare$

**Lemma A.7.** *There is a constant  $C = C(N)$  such that, if  $\psi_\epsilon \in H^1(\Omega)$  then,*

$$|T_\epsilon^0 \psi_\epsilon| + |T_\epsilon^1 \psi_\epsilon| \leq \begin{cases} C(2) |\ln \epsilon|^{1/2} \|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N = 2 \\ C \epsilon^{(2-N)/2} \|\psi_\epsilon\|_{H^1(\Omega)}, & \text{for } N > 2. \end{cases} \quad (\text{A.22})$$

**Proof:** The proof of this lemma follows from the previous and from the fact that  $|\widehat{T}_r^0 \psi_\epsilon| + |\widehat{T}_r^1 \psi_\epsilon| \leq C \|\psi_\epsilon\|_{H^1(\Omega)}$ .  $\blacksquare$

**A.5. Proof of Proposition 2.7.** In this Section we will provide a proof of Proposition 2.7.

We need to prove first some preliminary results.

Let  $f_\epsilon \in U_\epsilon^p$  and define the functions  $u_\epsilon \in H^1(\Omega_\epsilon)$ ,  $w_\epsilon \in H^1(\Omega)$  and  $v_\epsilon \in H^1(0, 1)$  as the solutions of the following linear elliptic problems:

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon &= f_\epsilon, & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} &= 0, & \text{in } \partial\Omega_\epsilon \end{cases} \quad (\text{A.23})$$

$$\begin{cases} -\Delta w_\epsilon + w_\epsilon &= f_\epsilon, & \text{in } \Omega \\ \frac{\partial w_\epsilon}{\partial n} &= 0, & \text{in } \partial\Omega \end{cases} \quad (\text{A.24})$$

$$\begin{cases} -\frac{1}{g}(g(v_\epsilon)_s)_s + v_\epsilon &= M_\epsilon f_\epsilon, & \text{in } (0, 1) \\ v_\epsilon(0) &= w_\epsilon(0), & v_\epsilon(1) = w_\epsilon(1). \end{cases} \quad (\text{A.25})$$

We have the following fundamental result

**Proposition A.8.** *Let  $p > N$ . There exists a constant  $C > 0$  such that for any  $f_\epsilon \in U_\epsilon^p$  with  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ , we have*

$$\|u_\epsilon - w_\epsilon\|_{H^1(\Omega)}^2 + \|u_\epsilon - v_\epsilon\|_{H^1(\mathbb{R}_\epsilon)}^2 \leq \begin{cases} C \epsilon^2 |\ln \epsilon|, & \text{for } N = 2 \\ C \epsilon^N, & \text{for } N > 2 \end{cases} \quad (\text{A.26})$$

**Proof:** Notice first that since  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ , we have that  $\|f_\epsilon\|_{L^p(\Omega)} \leq 1$ , which implies that, since  $p > N$ ,  $w_\epsilon \in W^{2,p}(\Omega) \hookrightarrow C^1(\bar{\Omega})$  and  $\|w_\epsilon\|_{C^1(\bar{\Omega})} \leq C$  with some constant  $C$  independent of  $\epsilon$ . With a similar regularity argument, we can easily show that  $v_\epsilon \in C^1([0, 1])$  and  $\|v_\epsilon\|_{C^1([0, 1])} \leq C$ , with  $C$  independent of  $\epsilon$ .



The solutions of the three problems (A.23), (A.24), (A.25) can be obtained by a minimization procedure. That is, if we define

$$\begin{aligned}\lambda_\epsilon &= \min_{\phi \in H^1(\Omega_\epsilon)} \left\{ \frac{1}{2} \int_{\Omega_\epsilon} (|\nabla \phi|^2 + \phi^2) dx - \int_{\Omega_\epsilon} f_\epsilon \phi dx \right\} \\ \mu_\epsilon &= \min_{\phi \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} (|\nabla \phi|^2 + \phi^2) dx - \int_{\Omega} f_\epsilon \phi dx \right\}\end{aligned}$$

$$\tau_\epsilon = \min \left\{ \frac{1}{2} \int_0^1 g|\phi'|^2 + g\phi^2 - \int_0^1 g\phi : \phi \in H^1(0,1), \phi(0) = w_\epsilon(p_0), \phi(1) = w_\epsilon(p_1) \right\},$$

then  $\lambda_\epsilon$ ,  $\mu_\epsilon$  and  $\tau_\epsilon$  are attained in  $u_\epsilon$ ,  $w_\epsilon$ ,  $v_\epsilon$  respectively, and only there.

Let us first find a relationship among the three values  $\lambda_\epsilon$ ,  $\mu_\epsilon$  and  $\tau_\epsilon$ .

If we take the function  $\varphi_\epsilon(x) = E_C^\epsilon(w_\epsilon, v_\epsilon)$ , and denote by  $\tilde{v}_\epsilon$  the component of  $\varphi_\epsilon$  in  $R_\epsilon$ , we obtain that

$$\begin{aligned}\lambda_\epsilon &\leq \frac{1}{2} \int_{\Omega_\epsilon} (|\nabla \varphi_\epsilon|^2 + \varphi_\epsilon^2) dx - \int_{\Omega_\epsilon} f_\epsilon \varphi_\epsilon dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla w_\epsilon|^2 + w_\epsilon^2) dx - \int_{\Omega} f_\epsilon w_\epsilon dx + \frac{1}{2} \int_{R_\epsilon} (|\nabla \tilde{v}_\epsilon|^2 + \tilde{v}_\epsilon^2) dx - \int_{R_\epsilon} f_\epsilon \tilde{v}_\epsilon dx \\ &= \mu_\epsilon + \frac{1}{2} \int_{R_\epsilon} (|\nabla \tilde{v}_\epsilon|^2 + \tilde{v}_\epsilon^2) dx - \int_{R_\epsilon} f_\epsilon \tilde{v}_\epsilon dx.\end{aligned}$$

It follows from Lemma A.4 (A.14) and (A.15)

$$\begin{aligned}\frac{1}{2} \int_{R_\epsilon} (|\nabla \tilde{v}_\epsilon|^2 + \tilde{v}_\epsilon^2) dx &\leq \frac{1}{2} \int_{R_\epsilon} (|\nabla v_\epsilon|^2 + v_\epsilon^2) dx + C\epsilon^N \|w_\epsilon\|_{C^1(\Omega)} \|v_\epsilon\|_{C^1(\Omega)} \\ &\leq \epsilon^{N-1} \frac{1}{2} \int_0^1 (g(s)|v'_\epsilon|^2 + g(s)v_\epsilon^2) ds + C\epsilon^N\end{aligned}$$

where we have used that  $\|w_\epsilon\|_{C^1(\Omega)}$  and  $\|v_\epsilon\|_{C^1(0,1)}$  are uniformly bounded. Moreover,

$$\begin{aligned}\int_{R_\epsilon} f_\epsilon \tilde{v}_\epsilon dx &= \int_{R_\epsilon} M_\epsilon(f_\epsilon) v_\epsilon dx + \int_{R_\epsilon} M_\epsilon(f_\epsilon) (\tilde{v}_\epsilon - v_\epsilon) dx \\ &= \epsilon^{N-1} \int_0^1 g(s) M_\epsilon(f_\epsilon) v_\epsilon ds + \int_{\tilde{R}_\epsilon} M_\epsilon(f_\epsilon) (\tilde{v}_\epsilon - v_\epsilon) dx,\end{aligned}$$

But,

$$\left| \int_{\tilde{R}_\epsilon} M_\epsilon(f_\epsilon) (\tilde{v}_\epsilon - v_\epsilon) dx \right| \leq \|M_\epsilon f_\epsilon\|_{L^2(\tilde{R}_\epsilon)} \|\tilde{v}_\epsilon - v_\epsilon\|_{L^2(\tilde{R}_\epsilon)} \leq C \|f_\epsilon\|_{L^2(\tilde{R}_\epsilon)} \epsilon^{\frac{N+2}{2}} \|w_\epsilon\|_{C^1(\bar{\Omega})}$$

and by Holder,

$$\|f_\epsilon\|_{L^2(\tilde{R}_\epsilon)} \leq \|f_\epsilon\|_{L^p(\tilde{R}_\epsilon)} |\tilde{R}_\epsilon|^{\frac{1}{2} - \frac{1}{p}} \leq \epsilon^{-\frac{N-1}{p}} \|f_\epsilon\|_{L^p(\tilde{R}_\epsilon)} \epsilon^{\frac{N-1}{p}} (\epsilon^N)^{\frac{1}{2} - \frac{1}{p}} \leq C \epsilon^{\frac{N}{2} - \frac{1}{p}}$$

This implies

$$\left| \int_{\tilde{R}_\epsilon} M_\epsilon(f_\epsilon) (\tilde{v}_\epsilon - v_\epsilon) dx \right| \leq C \epsilon^{N+1-\frac{1}{p}}$$

In particular, we obtain the following upper bounds for  $\lambda_\epsilon$ ,

$$\lambda_\epsilon \leq \mu_\epsilon + \epsilon^{N-1} \tau_\epsilon + C\epsilon^N. \quad (\text{A.27})$$

To obtain the lower bounds, we proceed as follows. From the definition of  $\lambda_\epsilon$  we have

$$\begin{aligned} \lambda_\epsilon &= \frac{1}{2} \int_{\Omega_\epsilon} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{\Omega_\epsilon} f_\epsilon u_\epsilon dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{\Omega} f_\epsilon u_\epsilon dx + \frac{1}{2} \int_{R_\epsilon} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{R_\epsilon} f_\epsilon u_\epsilon dx. \end{aligned} \quad (\text{A.28})$$

But

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{\Omega} f_\epsilon u_\epsilon dx &= \frac{1}{2} \int_{\Omega} (|\nabla u_\epsilon - \nabla w_\epsilon + \nabla w_\epsilon|^2 + (u_\epsilon - w_\epsilon + w_\epsilon)^2) dx \\ &\quad - \int_{\Omega} f_\epsilon (u_\epsilon - w_\epsilon + w_\epsilon) dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla w_\epsilon|^2 + w_\epsilon^2) dx - \int_{\Omega} f_\epsilon w_\epsilon dx + \frac{1}{2} \int_{\Omega} (|\nabla u_\epsilon - \nabla w_\epsilon|^2 + (u_\epsilon - w_\epsilon)^2) dx \\ &\quad + \int_{\Omega} (u_\epsilon - w_\epsilon) w_\epsilon + \int_{\Omega} (\nabla u_\epsilon - \nabla w_\epsilon) \nabla w_\epsilon - \int_{\Omega} f_\epsilon (u_\epsilon - w_\epsilon) \\ &= \frac{1}{2} \int_{\Omega} (|\nabla w_\epsilon|^2 + w_\epsilon^2) dx - \int_{\Omega} f_\epsilon w_\epsilon dx + \frac{1}{2} \int_{\Omega} (|\nabla u_\epsilon - \nabla w_\epsilon|^2 + (u_\epsilon - w_\epsilon)^2) dx \end{aligned} \quad (\text{A.29})$$

where in the last equality we have used the integration by parts of  $\int_{\Omega} (\nabla u_\epsilon - \nabla w_\epsilon) \nabla w_\epsilon$  and the fact that  $w_\epsilon$  is the solution of the elliptic problem (A.24) in  $\Omega$ , that is

$$\begin{aligned} \int_{\Omega} (\nabla u_\epsilon - \nabla w_\epsilon) \nabla w_\epsilon &= \int_{\partial\Omega} (u_\epsilon - w_\epsilon) \frac{\partial w_\epsilon}{\partial n} - \int_{\Omega} (u_\epsilon - w_\epsilon) \Delta w_\epsilon \\ &= \int_{\Omega} (u_\epsilon - w_\epsilon) (f_\epsilon - w_\epsilon). \end{aligned}$$

Also we have

$$\begin{aligned} \frac{1}{2} \int_{R_\epsilon} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{R_\epsilon} f_\epsilon u_\epsilon dx &= \frac{1}{2} \int_{R_\epsilon} (|\nabla u_\epsilon - \nabla v_\epsilon + \nabla v_\epsilon|^2 + (u_\epsilon - v_\epsilon + v_\epsilon)^2) dx \\ &\quad - \int_{R_\epsilon} f_\epsilon v_\epsilon dx - \int_{R_\epsilon} f_\epsilon (u_\epsilon - v_\epsilon) \\ &= \frac{1}{2} \int_{R_\epsilon} (|\nabla u_\epsilon - \nabla v_\epsilon|^2 + (u_\epsilon - v_\epsilon)^2) + \frac{1}{2} \int_{R_\epsilon} (|\nabla v_\epsilon|^2 + v_\epsilon^2) \\ &\quad + \int_{R_\epsilon} (\nabla u_\epsilon - \nabla v_\epsilon) \nabla v_\epsilon + \int_{R_\epsilon} (u_\epsilon - v_\epsilon) v_\epsilon \\ &\quad - \int_{R_\epsilon} f_\epsilon v_\epsilon dx - \int_{R_\epsilon} f_\epsilon (u_\epsilon - v_\epsilon). \end{aligned} \quad (\text{A.30})$$

But

$$\begin{aligned} \int_{R_\epsilon} (\nabla u_\epsilon - \nabla v_\epsilon) \nabla v_\epsilon &= \int_{R_\epsilon} \left( \frac{\partial u_\epsilon}{\partial s} - \frac{\partial v_\epsilon}{\partial s} \right) \frac{\partial v_\epsilon}{\partial s} \\ &= \int_{R_\epsilon} \left( \frac{\partial u_\epsilon}{\partial s} - \frac{\partial M_\epsilon u_\epsilon}{\partial s} \right) \frac{\partial v_\epsilon}{\partial s} + \int_{R_\epsilon} \left( \frac{\partial M_\epsilon u_\epsilon}{\partial s} - \frac{\partial v_\epsilon}{\partial s} \right) \frac{\partial v_\epsilon}{\partial s} = I_1 + I_2. \end{aligned} \quad (\text{A.31})$$

First we analyze  $I_1$ . Observe first that with the change of variables  $y = \varepsilon L_s(z)$ , (see (2.1) and (2.2) for the definition of  $L_s$  and  $L$ ), we get

$$\frac{1}{|\Gamma_\varepsilon^s|} \int_{\Gamma_\varepsilon^s} u_\varepsilon(s, y) dy = \int_{B(0,1)} u_\varepsilon(s, \varepsilon L_s(z)) \frac{1}{|\Gamma_1^s|} J_{L_s}(z) dz$$

This implies that

$$\begin{aligned} \frac{dM_\varepsilon u_\varepsilon}{ds}(s) &= \frac{d}{ds} \int_{B(0,1)} u_\varepsilon(s, \varepsilon L_s(z)) \frac{J_{L_s}(z)}{|\Gamma_1^s|} dz = \int_{B(0,1)} \frac{\partial u_\varepsilon}{\partial s}(s, \varepsilon L_s(z)) \frac{J_{L_s}(z)}{|\Gamma_1^s|} dz \\ &\quad + \int_{B(0,1)} \nabla_y u_\varepsilon(s, \varepsilon L_s(z)) \varepsilon \frac{\partial}{\partial s}(L_s(z)) \frac{J_{L_s}(z)}{|\Gamma_1^s|} dz \\ &\quad + \int_{B(0,1)} u_\varepsilon(s, \varepsilon L_s(z)) \frac{\partial}{\partial s}(J_{L_s}(z)/|\Gamma_1^s|) dz \\ &= K_1 + K_2 + K_3 \end{aligned}$$

Undoing the change of variables in  $K_1$ , we get

$$K_1 = \frac{1}{\varepsilon^{N-1}} \int_{\Gamma_\varepsilon^s} \frac{\partial u_\varepsilon}{\partial s}(s, y) dy$$

Moreover, using that  $\left| \frac{\partial L_s(z)}{\partial s} \right| \leq C$  and undoing the change of variables, we have

$$|K_2| \leq C\varepsilon \int_{B(0,1)} |\nabla_y u_\varepsilon(s, \varepsilon L_s(z))| J_{L_s}(z) dz \leq C \frac{\varepsilon}{|\Gamma_\varepsilon^s|} \int_{\Gamma_\varepsilon^s} |\nabla_y u_\varepsilon(s, y)| dy$$

Now,

$$\begin{aligned} K_3 &= \int_{B(0,1)} u_\varepsilon(s, \varepsilon L_s(z)) \frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z) dz \\ &= \int_{B(0,1)} (u_\varepsilon(s, \varepsilon L_s(z)) - (M_\varepsilon u_\varepsilon)(s)) \frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z) dz \\ &\quad + (M_\varepsilon u_\varepsilon)(s) \int_{B(0,1)} \frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z) dz \end{aligned}$$

But,

$$\int_{B(0,1)} \frac{\partial(J_{L_s}/|\Gamma_1^s|)}{\partial s}(z) dz = \frac{d}{ds} \left( \frac{1}{|\Gamma_1^s|} \int_{B(0,1)} J_{L_s}(z) dz \right) = 0$$

because  $\int_{B(0,1)} J_{L_s}(z) dz = |\Gamma_1^s|$ .

Similarly undoing the change of variables and using that  $\left| \frac{\partial(J_{L_s}(z)/|\Gamma_1^s|)}{\partial s} \right| \leq C$  we have

$$|K_3| \leq C \frac{1}{|\Gamma_\varepsilon^s|} \int_{\Gamma_\varepsilon^s} |u_\varepsilon(s, y) - M_\varepsilon u_\varepsilon(s)| dy$$

Putting all the estimates together, we get

$$\left| \frac{dM_\varepsilon u_\varepsilon}{ds}(s) - M_\varepsilon \left( \frac{\partial u_\varepsilon}{\partial s} \right) \right| \leq C \frac{1}{|\Gamma_\varepsilon^s|} \left( \int_{\Gamma_\varepsilon^s} |u_\varepsilon(s, y) - M_\varepsilon u_\varepsilon(s)| dy + \varepsilon \int_{\Gamma_\varepsilon^s} |\nabla_y u_\varepsilon(s, y)| dy \right)$$

Now,

$$|I_1| \leq C\varepsilon \|v'_\varepsilon\|_{L^2(R_\varepsilon)} \|\nabla_y u_\varepsilon\|_{L^2(R_\varepsilon)} + C \|v'_\varepsilon\|_{L^2(R_\varepsilon)} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(R_\varepsilon)}$$

But by Poincare inequality,

$$\|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(R_\varepsilon)} \leq C\varepsilon \|\nabla_y u_\varepsilon\|_{L^2(R_\varepsilon)}$$

which implies that

$$|I_1| \leq C\varepsilon \|v'_\varepsilon\|_{L^2(R_\varepsilon)} \|\nabla_y u_\varepsilon\|_{L^2(R_\varepsilon)}$$

But we obviously have uniform estimates of  $\|v_\varepsilon\|_{H^1(0,1)}$ . Hence, we have that

$$|I_1| \leq C\varepsilon^{\frac{N+1}{2}} \|\nabla_y u_\varepsilon\| \leq C\varepsilon^{N+1} + \frac{1}{4} \|\nabla_y u_\varepsilon\|_{L^2(R_\varepsilon)}^2$$

and observe that  $\|\nabla_y u_\varepsilon\|_{L^2(R_\varepsilon)}^2 \leq \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{L^2(R_\varepsilon)}^2$ , which implies

$$|I_1| \leq C\varepsilon^{N+1} + \frac{1}{4} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{L^2(R_\varepsilon)}^2.$$

On the other hand, observe that

$$\begin{aligned} I_2 &= \varepsilon^{N-1} \int_0^1 ((M_\varepsilon u_\varepsilon)' - v'_\varepsilon) g(s) v'_\varepsilon \\ &= -\varepsilon^{N-1} \int_0^1 (M_\varepsilon u_\varepsilon - v_\varepsilon) (g(s) v'_\varepsilon)' + [(M_\varepsilon u_\varepsilon - v_\varepsilon) (g(s) v'_\varepsilon)]_0^1 \\ &= -\varepsilon^{N-1} \int_0^1 (M_\varepsilon u_\varepsilon - v_\varepsilon) [g(s) v_\varepsilon - g(s) M_\varepsilon f_\varepsilon] + \varepsilon^{N-1} [(M_\varepsilon u_\varepsilon - v_\varepsilon) (g(s) v'_\varepsilon)]_0^1 \\ &= - \int_{R_\varepsilon} (u_\varepsilon - v_\varepsilon) v_\varepsilon + \int_{R_\varepsilon} (u_\varepsilon - v_\varepsilon) f_\varepsilon + \int_{R_\varepsilon} (M_\varepsilon u_\varepsilon - u_\varepsilon) f_\varepsilon + \varepsilon^{N-1} [(M_\varepsilon u_\varepsilon - v_\varepsilon) (g(s) v'_\varepsilon)]_0^1. \end{aligned} \tag{A.32}$$

From which we obtain

$$\begin{aligned} \int_{R_\varepsilon} (|\nabla u_\varepsilon|^2 + u_\varepsilon^2) dx - \int_{R_\varepsilon} f_\varepsilon u_\varepsilon dx &= \int_{R_\varepsilon} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) dx - \int_{\Omega} f_\varepsilon v_\varepsilon dx \\ &\quad + \int_{R_\varepsilon} (|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + (u_\varepsilon - v_\varepsilon)^2) dx + \kappa(\varepsilon) + \eta(\varepsilon) + I_1, \end{aligned} \tag{A.33}$$

where  $\kappa(\varepsilon) = \int_{R_\varepsilon} (M_\varepsilon u_\varepsilon - u_\varepsilon) f_\varepsilon$  and  $\eta(\varepsilon) = \varepsilon^{N-1} [(M_\varepsilon u_\varepsilon - v_\varepsilon) (g(s) v'_\varepsilon)]_0^1$ .

From (A.5) (Lemma A.3), we obtain that

$$\begin{aligned}
|\kappa(\epsilon)| &\leq \left| \int_{R_\epsilon} (u_\epsilon - M_\epsilon u_\epsilon) f_\epsilon \right| \leq \|u_\epsilon - M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)} \|f_\epsilon\|_{L^2(R_\epsilon)} \\
&\leq C\epsilon^{\frac{N-1}{2}} \|u_\epsilon - M_\epsilon u_\epsilon\|_{L^2(R_\epsilon)} \left( \epsilon^{\frac{1-N}{2}} \|f_\epsilon\|_{L^2(R_\epsilon)} \right) \\
&\leq C\epsilon^{\frac{N+1}{2}} \|\nabla_y u\|_{L^2(R_\epsilon)} \leq C\epsilon^{N+1} + \frac{1}{4} \|\nabla_y u\|_{L^2(R_\epsilon)}^2 \\
&\leq C\epsilon^{N+1} + \frac{1}{4} \|\nabla u - \nabla v_\epsilon\|_{L^2(R_\epsilon)}^2
\end{aligned} \tag{A.34}$$

where we have used that  $\epsilon^{\frac{1-N}{2}} \|f_\epsilon\|_{L^2(R_\epsilon)} \leq C\epsilon^{\frac{1-N}{p}} \|f_\epsilon\|_{L^p(R_\epsilon)} \leq C$ .

We estimate now  $\eta(\epsilon)$ . For this, note first that  $v_\epsilon(0) = w_\epsilon(0, 0)$  and  $v_\epsilon(1) = w_\epsilon(1, 0)$ . In particular,

$$|T_\epsilon^0(w_\epsilon - v_\epsilon)| = \frac{1}{|\Gamma_\epsilon^0|} \left| \int_{\Gamma_\epsilon^0} (w_\epsilon(0, y) - w_\epsilon(0, 0)) dy \right| \leq \frac{C}{|\Gamma_\epsilon^0|} \int_{\Gamma_\epsilon^0} |y| dy \|w_\epsilon\|_{C^1(\bar{\Omega}_0)} \leq C\epsilon \|w_\epsilon\|_{C^1(\bar{\Omega}_0)}$$

and similarly, we obtain

$$|T_\epsilon^1(w_\epsilon - v_\epsilon)| \leq C\epsilon \|w_\epsilon\|_{C^1(\bar{\Omega}_0)}$$

Hence,

$$\begin{aligned}
|\eta(\epsilon)| &\leq C\epsilon^{N-1} (|T_\epsilon^1(u_\epsilon) - v_\epsilon(1)| + |T_\epsilon^0(u_\epsilon) - v_\epsilon(0)|) \|v_\epsilon\|_{C^1([0,1])} \\
&\leq C\epsilon^{N-1} (|T_\epsilon^1(u_\epsilon - w_\epsilon)| + |T_\epsilon^1(w_\epsilon) - v_\epsilon(1)| + |T_\epsilon^0(u_\epsilon - w_\epsilon)| + |T_\epsilon^0(w_\epsilon) - v_\epsilon(0)|) \|v_\epsilon\|_{C^1([0,1])} \\
&\leq C\epsilon^{N-1} (|T_\epsilon^1(u_\epsilon - w_\epsilon)| + |T_\epsilon^0(u_\epsilon - w_\epsilon)|) \|v_\epsilon\|_{C^1([0,1])} + C\epsilon^N \|v_\epsilon\|_{C^1([0,1])} \|w_\epsilon\|_{C^1(\bar{\Omega}_0)}
\end{aligned}$$

But using the fact that  $\|v_\epsilon\|_{C^1([0,1])}$ ,  $\|w_\epsilon\|_{C^1(\bar{\Omega}_0)} \leq C$  and applying Lemma A.7 with  $\psi_\epsilon = u_\epsilon - w_\epsilon$ , we get

$$\begin{aligned}
|\eta(\epsilon)| &\leq C(N)\epsilon^{N-1} [|T_\epsilon^0(u_\epsilon - v_\epsilon)| + |T_\epsilon^1(u_\epsilon - v_\epsilon)|] + C\epsilon^N \\
&\leq C(\Theta_N(\epsilon))^{1/2} \|u_\epsilon - w_\epsilon\|_{H^1(\Omega)} + C\epsilon^N
\end{aligned}$$

where

$$\Theta_N(\epsilon) = \begin{cases} \epsilon^2 |\ln \epsilon|, & \text{for } N = 2 \\ \epsilon^N, & \text{for } N > 2. \end{cases}$$

and therefore

$$|\eta(\epsilon)| \leq C\Theta_N(\epsilon) + \frac{1}{4} \|u_\epsilon - w_\epsilon\|_{H^1(\Omega)}^2.$$

where we have used that  $\Theta_N(\epsilon) \geq \epsilon^N$ .

Putting together all the estimates, we obtain

$$\begin{aligned} \int_{R_\epsilon} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \int_{R_\epsilon} f_\epsilon u_\epsilon dx &\geq \int_{R_\epsilon} (|\nabla v_\epsilon|^2 + v_\epsilon^2) dx - \int_{\Omega} f_\epsilon v_\epsilon dx \\ &\quad - \frac{1}{2} \|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)}^2 - \frac{1}{2} \|u_\epsilon - w_\epsilon\|_{H^1(\Omega)}^2 - C(N) \Theta_N(\epsilon). \end{aligned} \quad (\text{A.35})$$

Thus,

$$\lambda_\epsilon \geq \mu_\epsilon + \epsilon^{N-1} \tau_\epsilon + \frac{1}{2} \|u_\epsilon - w_\epsilon\|_{H^1(\Omega)}^2 + \frac{1}{2} \|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)}^2 - C \Theta_N(\epsilon).$$

Since we have obtained that  $\lambda_\epsilon \leq \mu_\epsilon + \epsilon^{N-1} \tau_\epsilon + C \epsilon^N$ , and  $\epsilon^N \leq \Theta_N(\epsilon)$ , then

$$\|u_\epsilon - w_\epsilon\|_{H^1(\Omega)}^2 + \|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)}^2 \leq C \Theta_N(\epsilon).$$

This completes the proof of the proposition. ■

**Lemma A.9.** *Let  $\{f_\epsilon\}$  be a sequence such that  $f_\epsilon \in U_\epsilon^p$  and  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ . Then, there are functions  $f \in L^p(\Omega)$  and  $h \in L_g^p(0, 1)$  such that*

$$\int_{\Omega} f_\epsilon w_\epsilon dx \longrightarrow \int_{\Omega} f w dx$$

and

$$\frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} f_\epsilon v_\epsilon dx \longrightarrow \int_0^1 g(s) h(s) v(s) ds$$

whenever  $\|w_\epsilon - w\|_{L^{p'}(\Omega)} + \epsilon^{\frac{1-N}{p'}} \|v_\epsilon - v\|_{L^{p'}(R_\epsilon)} \xrightarrow{\epsilon \rightarrow 0} 0$ .

**Proof:** Note that  $\|f_\epsilon\|_{U_\epsilon} \leq 1$  means

$$\int_{\Omega} (f_\epsilon(x))^p dx + \frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} (f_\epsilon(x))^p dx \leq 1.$$

It follows that, there exists  $f \in L^p(\Omega)$  such that  $f_\epsilon \rightarrow f$  weakly in  $L^p(\Omega)$ . Hence

$$\int_{\Omega} f_\epsilon w_\epsilon dx \longrightarrow \int_{\Omega} f w dx$$

whenever  $\|w_\epsilon - w\|_{L^p(\Omega)} \rightarrow 0$ .

Also note that, writing  $x = (s, y)$  with  $s \in \mathbb{R}$ ,  $y \in \mathbb{R}^{N-1}$  and  $\epsilon \tilde{y} = y$ , we have that

$$\frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} (f_\epsilon(x))^p dx \leq \int_{R_1} (\tilde{f}_\epsilon(s, \tilde{y}))^p ds d\tilde{y} \leq 1$$

where  $\tilde{f}_\epsilon(s, \tilde{y}) = f_\epsilon(s, \epsilon \tilde{y})$ . So, there exists  $\tilde{h} \in L^p(R_1)$  such that  $\tilde{f}_\epsilon \rightarrow \tilde{h}$  weakly in  $L^p(R_1)$ .

Let us show that  $\tilde{h}$  is independent of  $y$ . Note that, if  $\tilde{\phi} \in C_0^\infty(R_1)$  then,

$$\int_{R_1} \tilde{f}_\epsilon(s, \tilde{y}) \frac{\partial \tilde{\phi}}{\partial \tilde{y}_i}(s, \tilde{y}) ds d\tilde{y} = \frac{\epsilon}{\epsilon^{N-1}} \int_{R_\epsilon} f_\epsilon(s, y) \frac{\partial \phi}{\partial y_i}(s, y) ds dy \xrightarrow{\epsilon \rightarrow 0} 0.$$

It follows that, for all  $\tilde{\phi} \in C_0^\infty(R_1)$

$$\int_{R_1} \tilde{h} \frac{\partial \tilde{\phi}}{\partial \tilde{y}_i} dx = 0.$$

Hence,  $\tilde{h}(s, y) = g(s)h(s)$  for some  $h \in L^2(0, 1)$ . Furthermore, if  $\epsilon^{\frac{1-N}{p}} \|v_\epsilon - v\|_{L^p(R_\epsilon)} \rightarrow 0$  then,

$$\frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} f_\epsilon(x) v_\epsilon(x) dx \longrightarrow \int_0^1 g(s) h(s) v(s) ds.$$

■

Now we show the following result,

**Proposition A.10.** *Let  $p > N$  and consider a sequence  $f_\epsilon \in U_\epsilon^p$  with  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ . Let  $(f, h) \in L^p(\Omega) \times L_g^p(0, 1)$  such that  $f_\epsilon \rightarrow (f, h)$  weakly in the sense of Lemma A.9. Then*

$$\|u_\epsilon - w\|_{H^1(\Omega)} + \frac{1}{\epsilon^{(N-1)/2}} \|u_\epsilon - v\|_{H^1(R_\epsilon)} \rightarrow 0 \quad (\text{A.36})$$

where  $u_\epsilon$ ,  $w$  and  $v$  are the solutions of the following problems,

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon &= f_\epsilon, & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} &= 0, & \text{in } \partial\Omega_\epsilon, \end{cases} \quad \begin{cases} -\Delta w + w &= f, & \text{in } \Omega \\ \frac{\partial w}{\partial n} &= 0, & \text{in } \partial\Omega \end{cases} \quad (\text{A.37})$$

$$\begin{cases} -\frac{1}{g}(gv_s)_s + v &= h, & \text{in } (0, 1) \\ v(0) &= w(0), & v(1) = w(1). \end{cases} \quad (\text{A.38})$$

**Proof:** If  $w_\epsilon$  and  $v_\epsilon$  are given by (A.24) and (A.25) respectively, and taking into account that  $p > N$  and  $f_\epsilon \rightarrow f$  weakly in  $L^p(\Omega)$  we easily obtain that  $\|w_\epsilon - w\|_{C^1(\bar{\Omega})} \rightarrow 0$ , which also implies that  $\|w_\epsilon - w\|_{H^1(\Omega)} \rightarrow 0$ . From Lemma A.8 we get that

$$\|u_\epsilon - w\|_{H^1(\Omega)} \rightarrow 0.$$

Moreover, since  $w_\epsilon \rightarrow w$  in  $C^0(\bar{\Omega})$  and  $M_\epsilon f_\epsilon \rightarrow h$  weakly in  $L_g^p(0, 1)$  it is very simple to see that we have

$$\|v_\epsilon - v\|_{H^1(0,1)} \rightarrow 0$$

which implies that

$$\frac{1}{\epsilon^{N-1}} \|v_\epsilon - v\|_{H^1(R_\epsilon)}^2 \rightarrow 0$$

Hence, with this last statement and using Lemma A.8 we get

$$\frac{1}{\epsilon^{(N-1)/2}} \|u_\epsilon - v\|_{H^1(R_\epsilon)} \leq \frac{1}{\epsilon^{(N-1)/2}} (\|u_\epsilon - v_\epsilon\|_{H^1(R_\epsilon)} + \|v_\epsilon - v\|_{H^1(R_\epsilon)}) \rightarrow 0$$

which proves the result. ■

We obtain now a result on uniform  $L^\infty(\Omega_\epsilon)$  bounds for the family of solutions  $\{u_\epsilon\}_{\epsilon \in [0,1]}$  of the problem (A.23). This result will show part *i*) of Proposition 2.7.

**Lemma A.11.** *There exists a constant  $C$  independent of  $\epsilon$  such that for all  $f_\epsilon \in U_\epsilon^p$  with  $p > N/2$  and  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$  if  $u_\epsilon$  is the solution of (A.23) then*

$$\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C \quad (\text{A.39})$$

**Proof:** Let us define the functions  $u_\epsilon^1$  and  $u_\epsilon^2$  as the solutions of the following problems

$$\begin{cases} -\Delta u_\epsilon^1 + u_\epsilon^1 = f_\epsilon|_{R_\epsilon}, & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon^1}{\partial n} = 0, & \text{in } \partial\Omega_\epsilon \end{cases} \quad (\text{A.40})$$

and

$$\begin{cases} -\Delta u_\epsilon^2 + u_\epsilon^2 = f_\epsilon|_\Omega, & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon^2}{\partial n} = 0, & \text{in } \partial\Omega_\epsilon \end{cases} \quad (\text{A.41})$$

We proceed in several steps.

**Step 1.** Let us show that for each compact set  $K \subset \bar{\Omega} \setminus \{0, 1\}$  there is a constant  $C$  independent of  $\epsilon$  such that

$$\|u_\epsilon\|_{L^\infty(K)} \leq \|u_\epsilon^1\|_{L^\infty(K)} + \|u_\epsilon^2\|_{L^\infty(K)} \leq C.$$

This follows easily with a cutoff function and an elementary bootstrap argument.

**Step 2.** Let us see now that there is a constant  $C$  independent of  $\epsilon$  such that

$$\|u_\epsilon^1\|_{L^\infty(R_\epsilon)} \leq C\epsilon^{\frac{1-N}{p}} \|f\|_{L^p(R_\epsilon)} \leq \|f\|_{U_\epsilon^p}.$$

To prove this result first note that, from Step 1, for any compact subset of  $\bar{\Omega} \setminus \{0, 1\}$  we have that  $u_\epsilon^1$  is uniform bounded in  $L^\infty(K)$ . Hence, if for a small fixed  $\delta > 0$ , we define  $\tilde{R}_\epsilon = ([-\delta, 0] \times \Gamma_\epsilon^0) \cup R_\epsilon \cup ([1, 1+\delta] \times \Gamma_\epsilon^1)$ , then, there exists a  $k > 0$  such that  $|u_\epsilon^1| \leq k$  in  $\Gamma_{(-\delta, 1+\delta)} = (\{-\delta\} \times \Gamma_\epsilon^0) \cup (\{1+\delta\} \times \Gamma_\epsilon^1)$

Hence, if we define  $\phi_\epsilon = (u_\epsilon^1 - k)^+$  in  $\Omega_\epsilon$ , after multiplying the equation by  $\phi_\epsilon$  and integrating by parts we have that

$$\int_{\tilde{R}_\epsilon} |\nabla \phi_\epsilon|^2 + \phi_\epsilon^2 \leq \int_{\Omega_\epsilon} |\nabla \phi_\epsilon|^2 + \phi_\epsilon^2 = \int_{R_\epsilon} (f_\epsilon - k)\phi_\epsilon \leq \int_{R_\epsilon} |f_\epsilon|\phi_\epsilon \quad (\text{A.42})$$

Writing  $x = (s, y)$  with  $s \in \mathbb{R}$ ,  $y \in \mathbb{R}^{N-1}$  and changing the variables  $(s, y)$  to  $(s, \tilde{y})$  where to  $\epsilon\tilde{y} = y$  we obtain from (A.42) that

$$\int_{\tilde{R}_1} |\nabla \tilde{\phi}_\epsilon|^2 + |\tilde{\phi}_\epsilon|^2 \leq \int_{R_1} \tilde{f}_\epsilon \tilde{\phi}_\epsilon. \quad (\text{A.43})$$

Where  $\tilde{\phi}_\epsilon(s, \tilde{y}) = \phi_\epsilon(s, \epsilon\tilde{y})$  and  $\tilde{f}_\epsilon(s, \tilde{y}) = f_\epsilon(s, \epsilon\tilde{y})$ . Proceeding exactly as in Lemma B.1 iii) of [8] we obtain that

$$\|\tilde{\phi}_\epsilon\|_{H^1(\tilde{R}_1)}^2 \leq \|\tilde{f}_\epsilon\|_{L^p(R_1)} \|\tilde{\phi}_\epsilon\|_{L^{p'}(R_1)} \leq \|\tilde{f}_\epsilon\|_{L^p(R_1)} \|\tilde{\phi}_\epsilon\|_{L^{\frac{2N}{N-2}}(R_1)} |A_k|^{\frac{1}{p'} + \frac{1}{N} - \frac{1}{2}} \quad (\text{A.44})$$

where  $A_k = \{(x, \tilde{y}) \in \tilde{R}_1 : u_\epsilon^1 > k\}$ . From this we have

$$\|\tilde{\phi}_\epsilon\|_{H^1(\tilde{R}_1)} \leq \|\tilde{f}_\epsilon\|_{L^p(R_1)} |A_k|^{\frac{1}{p'} + \frac{1}{N} - \frac{1}{2}}. \quad (\text{A.45})$$

From (A.44) and (A.45) we have that

$$\|\tilde{\phi}_\epsilon\|_{L^1(\tilde{R}_1)} \leq \|\tilde{\phi}_\epsilon\|_{L^{\frac{2N}{N-2}}(\tilde{R}_1)} |A_k|^{\frac{N+2}{2N}} \leq C \|\tilde{f}_\epsilon\|_{H^1(\tilde{R}_1)} |A_k|^{\frac{N+2}{2N}} \leq C \|\tilde{f}_\epsilon\|_{L^p(R_1)} |A_k|^{\frac{1}{p'} + \frac{2}{N}}.$$



Since, for  $p > \frac{N}{2}$  we have that  $\frac{1}{p'} + \frac{2}{N} > 1$ , it follows from Lemma 5.1 in [34] that

$$\|\phi_\epsilon\|_{L^\infty(\tilde{R}_\epsilon)} = \|\tilde{\phi}_\epsilon\|_{L^\infty(\tilde{R}_1)} \leq C \|\tilde{f}\|_{L^p(R_1)} = C \epsilon^{\frac{1-N}{p}} \|f_\epsilon\|_{L^p(R_\epsilon)}$$

with  $C = C(R_1, N, p)$ .

**Step 3.** We show that  $\|u_\epsilon^2\|_{L^\infty(\Omega)} \leq C$ .

After multiplying the equation (A.41) by  $\psi_\epsilon = (u_\epsilon^2 - k)^+$ ,  $k > 0$ , and integrating by parts we have that

$$\int_\Omega |\nabla \psi_\epsilon|^2 + |\psi_\epsilon|^2 \leq \int_{\Omega_\epsilon} |\nabla \psi_\epsilon|^2 + |\psi_\epsilon|^2 = \int_{\Omega_\epsilon} (f_\epsilon - k) \psi_\epsilon \leq \int_\Omega f_\epsilon \psi_\epsilon. \quad (\text{A.46})$$

Proceeding exactly as in Lemma B.1 iii) of [8] we obtain that

$$\|\psi_\epsilon\|_{H^1(\Omega)}^2 \leq \|f_\epsilon\|_{L^p(\Omega)} \|\psi_\epsilon\|_{L^{p'}(\Omega)} \leq \|f_\epsilon\|_{L^p(\Omega)} \|\psi_\epsilon\|_{L^{\frac{2N}{N-2}}(\Omega)} |A_k|^{\frac{1}{p'} + \frac{1}{N} - \frac{1}{2}} \quad (\text{A.47})$$

where  $A_k = \{(x, \tilde{y}) \in \Omega : u_\epsilon^2 > k\}$ . From this we have

$$\|\psi_\epsilon\|_{H^1(\Omega)} \leq \|f_\epsilon\|_{L^p(\Omega)} |A_k|^{\frac{1}{p'} + \frac{1}{N} - \frac{1}{2}}. \quad (\text{A.48})$$

From (A.47) and (A.48) we have that

$$\|\psi_\epsilon\|_{L^1(\Omega)} \leq \|\psi_\epsilon\|_{L^{\frac{2N}{N-2}}(\Omega)} |A_k|^{\frac{N+2}{2N}} \leq C \|\psi_\epsilon\|_{H^1(\Omega)} |A_k|^{\frac{N+2}{2N}} \leq C \|f_\epsilon\|_{L^p(\Omega)} |A_k|^{\frac{1}{p'} + \frac{2}{N}}.$$

Since, for  $p > \frac{N}{2}$  we have that  $\frac{1}{p'} + \frac{2}{N} > 1$ , it follows from Lemma 5.1 in [34] that

$$\|u_\epsilon^2\|_{L^\infty(\Omega)} = \|\tilde{\phi}_\epsilon\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)} = C \|f_\epsilon\|_{L^p(\Omega)}$$

with  $C = C(\Omega, N, p)$ .

**Step 4.** We show that  $\|u_\epsilon^1\|_{L^\infty(\Omega)} \leq C$  and  $\|u_\epsilon^2\|_{L^\infty(R_\epsilon)} \leq C$ . Observe that by the maximum principle  $\|u_\epsilon^1\|_{L^\infty(\Omega)} \leq \|u_\epsilon^1\|_{L^\infty(\Gamma_\epsilon^1 \cup \Gamma_\epsilon^0)}$  and  $\|u_\epsilon^2\|_{L^\infty(R_\epsilon)} \leq \|u_\epsilon^2\|_{L^\infty(\Gamma_\epsilon^1 \cup \Gamma_\epsilon^0)}$ , which both are bounded uniformly in  $\epsilon$  by the previous steps.  $\blacksquare$

We are in a position now to provide a complete proof of Proposition 2.7

**Proof of Proposition 2.7:** Observe that (1) follows directly from Lemma A.11.

If  $\|f_\epsilon\|_{U_\epsilon^p} \leq 1$ , then by Lemma A.9 we can get a subsequence, that we denote by  $\epsilon$  again, and  $(f, h) \in U_0^p$ , such that if  $u_\epsilon$  and  $(w, v)$  are given by (2.13) and (2.14), then (A.36) holds. In particular, this shows i). Moreover, from (A.36) we have

$$\|u_\epsilon - w\|_{L^2(\Omega)} + \frac{1}{\epsilon^{(N-1)/2}} \|u_\epsilon - v\|_{L^2(R_\epsilon)} \rightarrow 0.$$

Hölder's inequality implies

$$\|u_\epsilon - w\|_{L^1(\Omega)} + \frac{1}{\epsilon^{(N-1)}} \|u_\epsilon - v\|_{L^1(R_\epsilon)} \rightarrow 0. \quad (\text{A.49})$$

Since by Lemma A.11 we have  $\|u_\epsilon - w\|_{L^\infty(\Omega)} + \|u_\epsilon - v\|_{L^\infty(\Omega)} \leq C$ , interpolating this estimate with (A.49), we obtain,

$$\|u_\epsilon - w\|_{L^q(\Omega)} + \frac{1}{\epsilon^{(N-1)/q}} \|u_\epsilon - v\|_{L^q(R_\epsilon)} \rightarrow 0, \quad 1 \leq q < \infty.$$

This shows *ii*). Finally, *iii*) is proved with *i*) and *ii*) and using a standard cutoff and bootstrap procedure.

The last part of the proposition, statement (3), follows using (2), Lemma A.9 and a standard argument by contradiction. ■

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